<u>Definition</u>: The exponential map is the map $\mathcal{V}_\varepsilon \longrightarrow TM$.

$$(q, v) \longmapsto \gamma(1, q, v)$$

where $\gamma, \varepsilon, \mathcal{V}$ are as in Prop. 2.7.

Note $\gamma(1, q, v) = \gamma\left(|v|, q, \frac{v}{|v|}\right)$

In most applications we fix a point $p \in M$ and consider:

$$\begin{tikzcd} (\exp)_p : B_\varepsilon(0) \arrow[r] \arrow[rr, bend right] \arrow[d, phantom, "\cap"] & TM \arrow[r, "\pi"] & M \\ T_p M & & \end{tikzcd}$$

$\pi(p, v) = p$.

<u>Prop.</u>: For all $p \in M$, $\exists\ \varepsilon > 0$ s.t. $(\exp)_p : B_\varepsilon(0) \longrightarrow M$ is a diffeom. onto its image.

<u>Proof</u>: compute $\left(d(\exp)_p\right)_0 (v)$: choose a curve in $B_\varepsilon(0)$ through 0 with velocity vector v.

in fact choose $c(t) = tv$.

$$\frac{d}{dt}\left((\exp_p)(tv)\right)\Big|_{t=0} = \frac{d}{dt}\left(\gamma(1,p,tv)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\gamma(t,p,v)\right)\Big|_{t=0} = v$$

So $\left(d(\exp_p)\right)_0 = \mathrm{Id}$

So $(\exp)_p$ is a diffeo. on some neighborhood of 0. $\square$

<u>Examples</u>: (1) $\mathbb{R}^n$: geodesics are lines parametrized by length.

$T_v\mathbb{R}^n \doteq \mathbb{R}^n \qquad T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

$$\gamma : \,]-\delta,\delta[\, \times V \times B_\varepsilon(0) \longrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$$

$$(t, p, v) \longmapsto (p+tv, v)$$

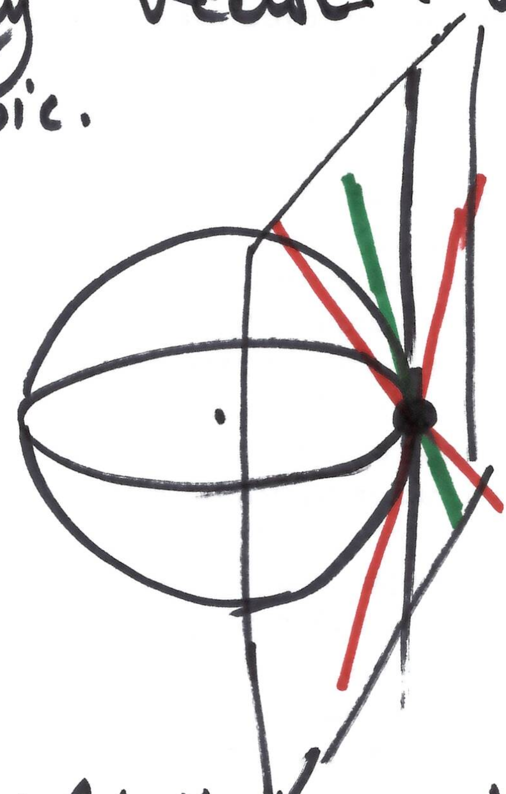
Note γ is well-defined on
$$\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$

(2) $S^n \subset \mathbb{R}^{n+1}$

Homework: the great circles are geodesics.

Any geodesic is a piece of a great circle because: $\forall p \in M$, $\forall v \in T_pM$ $\exists$ neighborhood V of p, and a unique geodesic in V through p with velocity vector v at time 0.

Through any point of S^n, there is always a unique great circle with given velocity vector: this is the unique geodesic.



lines are mapped onto great circles by $(\exp)_p$

Can explicitely write the maps.

$(\exp)_p$ is well-defined everywhere, a diffeo. (onto its image) on the ball of radius π.

Length minimizing properties of geodesics

Def: A piecewise differentiable curve is a continuous map
$c : [a, b] \longrightarrow M$
s.t. $\exists$ $t_1 < \dots < t_n \in [a, b]$.
s.t. $c|_{[t_i, t_{i+1}]}$ is differentiable $\forall i$ $\begin{pmatrix} t_0 = a \\ t_{n+1} = b \end{pmatrix}$

(means $\exists$ $(d, e) \supset [t_i, t_{i+1}]$
and an extension $c' : (d, e) \longrightarrow M$
of c which is differentiable)

We will show that $\forall$ $p \in M$,
$\exists$ neighborhood V of p in M s.t.

$\forall$ points $q, r \in V$
$\exists$ geodesic in V from q to r
and ~~that~~ any other piecewise diff curve from q to r is longer than the geodesic.

__Definition__ (A parametrized surface)
A parametrized surface is a differentiable map (∃ extension to an open set which is differentiable)
$$s : A \longrightarrow M$$
where $A \subset \mathbb{R}^2$
$\exists \quad U \subset \mathbb{R}^2$ s.t.
$$U \subset A \subset \overline{U}$$
and the boundary of A is a piecewise differentiable curve. (whose angles are $\neq \pi$)

__Remark:__ the angle θ of two vectors $v, w \in T_pM$ is defined via

$$\cos\theta = \frac{\langle v, w\rangle}{|v|\cdot|w|}$$

Use an orthonormal basis of T_pM.

A vector field on A is a map

$$V : A \longrightarrow TM$$
$$p \longmapsto V(p).$$

Such a vector field is differentiable if $\forall f$ diff. function on M, $V(f)$ is diff. on A.

(extends to a diff. function on some open $\mathbb{R}^2 \supset W \supset A$)

Choose coordinates (u, v) on $\mathbb{R}^2$

$\rightsquigarrow$ vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ on $\mathbb{R}^2$

$ds\left(\frac{\partial}{\partial u}\right)$, $ds\left(\frac{\partial}{\partial v}\right)$ are vector fields on $A \overset{s}{\hookrightarrow} M$.

In coordinates: $\mathbf{x} : U \longrightarrow M$, $U \subset \mathbb{R}^1$, $A \xrightarrow{s} M$.

$$\mathcal{X}^{-1}(s(u,v)) = (x_1(u,v), \dots, x_n(u,v))$$

$$d(\mathcal{X}^{-1}s)\left(\frac{\partial}{\partial u}\right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i}$$

$$d(\mathcal{X}^{-1}s)\left(\frac{\partial}{\partial v}\right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial v} \frac{\partial}{\partial x_i}$$

Notation: $ds\left(\frac{\partial}{\partial u}\right) =: \frac{\partial s}{\partial u}$ .....

Consequence: Symmetry lemma:

Notation as above, suppose M is differentiable with a symmetric conn..

Then $$\frac{D}{\partial v}\left(\frac{\partial s}{\partial u}\right) = \frac{D}{\partial u}\left(\frac{\partial s}{\partial v}\right)$$

Proof: use the above coordinates:

$$\frac{\partial s}{\partial u} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i}$$

$$\frac{D}{\partial v}\left(\frac{\partial s}{\partial u}\right) = \nabla_{\partial/\partial v}\left(\frac{\partial s}{\partial u}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 x_i}{\partial v \partial u} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \nabla_{\partial/\partial v}\left(\frac{\partial}{\partial x_i}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 x_i}{\partial v \partial u} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} \nabla_{\partial/\partial x_j}\left(\frac{\partial}{\partial x_i}\right)$$