$$\mathcal{X}^{-1}(s(u,v)) = (x_1(u,v), \dots, x_n(u,v))$$

$$d(\mathcal{X}^{-1}s)\left(\frac{\partial}{\partial u}\right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i}$$

$$d(\mathcal{X}^{-1}s)\left(\frac{\partial}{\partial v}\right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial v} \frac{\partial}{\partial x_i}$$

<u>Notation</u>: $ds\left(\frac{\partial}{\partial u}\right) =: \frac{\partial s}{\partial u}$ .....

<u>Consequence</u>: Symmetry lemma:

Notation as above, suppose M is differentiable with a <u>symmetric conn.</u>.

Then $$\frac{D}{\partial v}\left(\frac{\partial s}{\partial u}\right) = \frac{D}{\partial u}\left(\frac{\partial s}{\partial v}\right)$$

<u>Proof</u>: use the above coordinates:

$$\frac{\partial s}{\partial u} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i}$$

$$\frac{D}{\partial v}\left(\frac{\partial s}{\partial u}\right) = \nabla_{\partial/\partial v}\left(\frac{\partial s}{\partial u}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 x_i}{\partial v\, \partial u} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \nabla_{\partial/\partial v}\left(\frac{\partial}{\partial x_i}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 x_i}{\partial v\, \partial u} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} \nabla_{\partial/\partial x_j}\left(\frac{\partial}{\partial x_i}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 x_i}{\partial u \partial v} \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial u} \nabla_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_j}\right)$$

because $\nabla_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_j}\right) - \nabla_{\partial/\partial x_j}\left(\frac{\partial}{\partial x_i}\right)$

$$= \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0 \qquad \square$$

Two main results for geodesics (M Riemannian):

The local result: (Prop. 3.6):

$p \in M$, $U \subset M$

U a <u>normal</u> <u>neighborhood</u> of p

$B \subset U$ a <u>normal</u> <u>ball</u> centered at p.

$\gamma : [0,1] \longrightarrow B$ a geodesic with $\gamma(0) = p$

Then, for all piecewise diff. curves,

$c : [0,1] \longrightarrow M$ s.t. $c(0) = p$

and $c(1) = \gamma(1)$,

$\ell(c) \geq \ell(\gamma)$ with equality iff

$c([0,1]) = \gamma([0,1])$

Def: $p \in M$, $U \subset M$ is called a normal neighborhood of p if
$$U = \exp_p(V) \quad \text{where } V \subset T_pM$$
is an open set where $\exp_p$ is a diffeomorphism.

A normal ball centered at p is the image of $B_\varepsilon(0)$ for some $\varepsilon > 0$ s.t. $\exp_p$ is a diffeo. on $B_\varepsilon(0)$.
We usually denote
$$B_\varepsilon(p) := \exp_p(B_\varepsilon(0))$$
and ε is called the radius of $B_\varepsilon(p)$.

The global result:

Suppose $\gamma : [0,1] \longrightarrow M$ is a piecewise differentiable curve with constant speed (i.e., the parameter on γ is proportional to its length).

If the length of γ is less than or equal to the length of any other piecewise differentiable curve from $\gamma(0)$ to $\gamma(1)$, then γ is a geodesic. In particular, γ is differentiable or regular.

<u>Proof of the local result:</u>

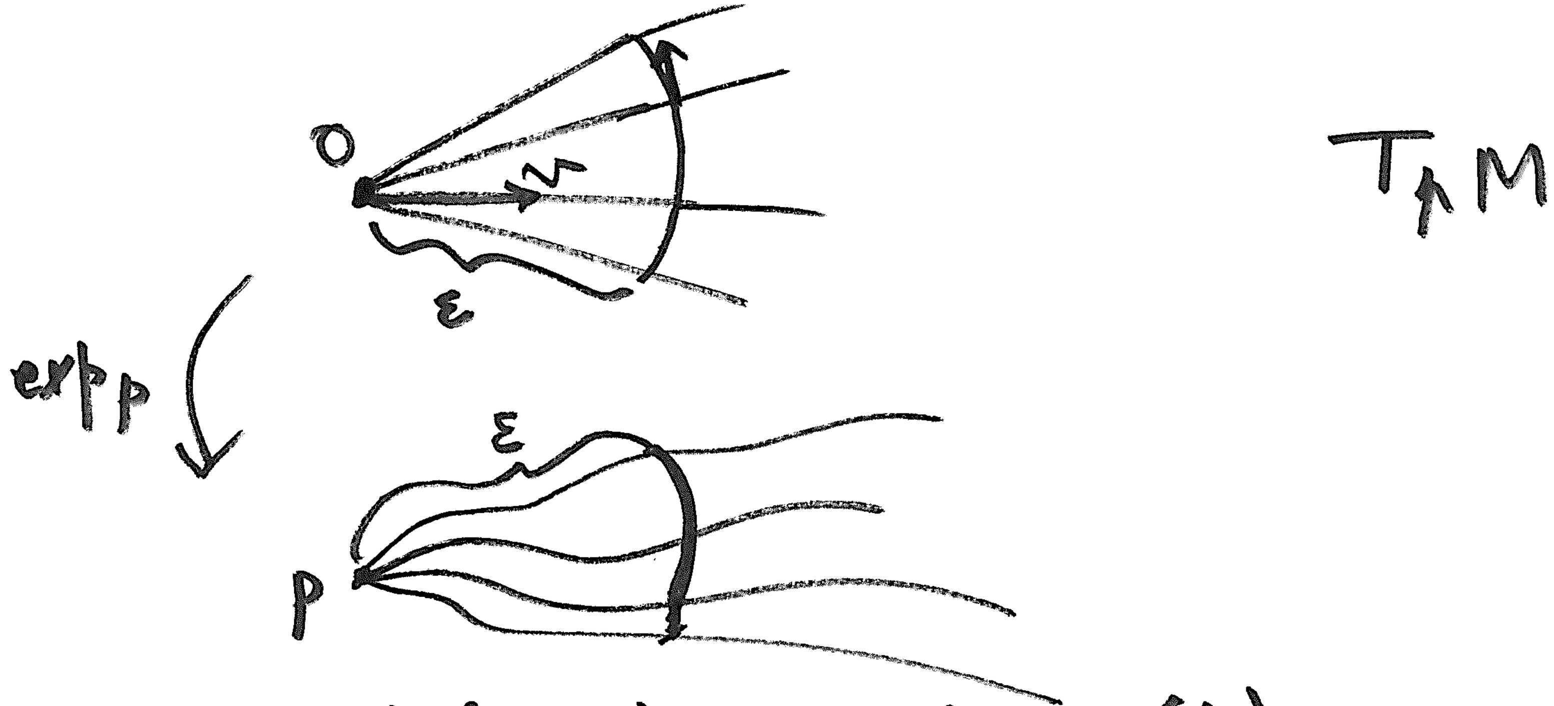
First assume $c([0,1]) \subset B$.

$$l(c) = \int_0^1 \left| \frac{dc}{dt} \right| dt$$

<u>recall</u>: $\exp_p(v) = \gamma(1, p, v)$.

$$\exp_p(tv) = \gamma(1, p, tv) = \gamma(t, p, v).$$

This shows that the radial line through 0 with direction v maps to the geodesic with velocity v at $t=0$.



$$\exp_p^{-1}(c(t)) = r(t)\,v(t).$$

where $\quad r : (0,1] \longrightarrow \mathbb{R}$

$\qquad\qquad v : (0,1] \longrightarrow \cancel{B}\, T_pM.$

and $\quad |v(t)| = 1 \quad \forall\, t.$

$r,\ v$ piecewise differentiable.

Define $f : (0,1] \times (0,1] \longrightarrow B.$

$$(r,\ t) \longmapsto \exp_p(r\,v(t))$$

Then
$$c(t) = \exp_p(r(t)\,v(t)) = f(r(t),\ t)$$

$$\frac{dc}{dt} = \frac{\partial f}{\partial r}\, r'(t) + \frac{\partial f}{\partial t}$$

Claim: $\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = 0.$

Admit this for the moment:

then $\left|\frac{dc}{dt}\right|^2 = \left|\frac{\partial f}{\partial r}\right|^2 (r'(t))^2 + \left|\frac{\partial f}{\partial t}\right|^2.$

$$\frac{\partial f}{\partial r} = (d(\exp_p))_{rv(t)}(v(t))$$

$$\left|\frac{\partial f}{\partial r}\right| = \left| d(\exp_p)_{rv(t)}(v(t)) \right|$$

$$= |v(t)| = 1$$

$$\frac{\partial f}{\partial t} = (d(\exp_p))_{rv(t)}(r v'(t))$$

$$\ell(c) = \lim_{\eta \to 0} \int_\eta^1 \left|\frac{dc}{dt}\right| dt$$

$$\int_\eta^1 \left|\frac{dc}{dt}\right| dt \geqslant \int_\eta^1 \left|\frac{\partial f}{\partial r}\right| r'(t)\, dt$$

$$= \int_\eta^1 |v(t)| r'(t)\, dt = \int_\eta^1 r'(t)\, dt.$$

$$= r(1) - r(\eta). = \ell(\gamma) - r(\eta)$$

$\eta \to 0$. get $\ell(c) \geqslant \ell(\gamma)$.

If $\quad \ell(c) = \ell(\gamma)$, then $\frac{\partial f}{\partial t} = 0$.

$\Rightarrow v'(t) = 0. \quad \forall t.$

$\Rightarrow v$ is constant.

$\Rightarrow (\exp_p^{-1} \circ c)([0,1])$ is a line through the origin.

~~$\Rightarrow c$ is a geodesic~~

$\cancel{\Rightarrow c = \gamma} \Rightarrow c([0,1]) = \gamma([0,1]) \quad \square$

<u>Proof of claim:</u> $\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = 0$.

$$\frac{\partial}{\partial r} \langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle =$$

$$\langle \frac{D}{\partial r}\left(\frac{\partial f}{\partial r}\right), \frac{\partial f}{\partial t} \rangle + \langle \frac{\partial f}{\partial r}, \frac{D}{\partial r}\frac{\partial f}{\partial t} \rangle$$

$\frac{D}{\partial r}\left(\frac{\partial f}{\partial r}\right) =$ covariant derivative of the velocity vector of geodesic

$= 0.$

Now use the symmetry lemma:

$$\langle \frac{\partial f}{\partial r}, \frac{D}{Dr} \frac{\partial f}{\partial t} \rangle = \langle \frac{\partial f}{\partial r}, \frac{D}{Dt} \frac{\partial f}{\partial r} \rangle$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r}\right\rangle \Rightarrow 0$$

~~length of velocity vector of a geodesic is constant.~~

$$= \frac{1}{2}\frac{\partial}{\partial t}\,|v(t)| = 0$$

So $\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}\right\rangle$ is independent of r.

plug in $r=0$: we are at the origin of T_pM or at p.

$$\frac{\partial f}{\partial r} = v \qquad \frac{\partial f}{\partial t} = 0$$

(take ~~the~~ limit when $r \to 0$) $\square$