\[ x^{-1}(s(u,v)) = (x_1(u,v), \ldots, x_n(u,v)) \]

\[ d(x^{-1}) \left( \frac{\partial}{\partial u} \right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i} \]

\[ d(x^{-1}) \left( \frac{\partial}{\partial v} \right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial v} \frac{\partial}{\partial x_i} \]

\underline{Notation:} \quad ds \left( \frac{\partial}{\partial u} \right) = \frac{\partial s}{\partial u}

\underline{Consequence:} \quad \text{Symmetry Lemma:}

\underline{Notation as above, suppose} \ M \ \text{is differentiable with a symmetric conn.}

\underline{Then} \quad \frac{D}{\partial s} \left( \frac{\partial s}{\partial u} \right) = \frac{D}{\partial u} \left( \frac{\partial s}{\partial v} \right)

\underline{Proof:} \quad \text{use the above coordinates:}

\[ \frac{\partial s}{\partial u} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i} \]

\[ \frac{D}{\partial s} \left( \frac{\partial s}{\partial u} \right) = \nabla_{\partial s} \left( \frac{\partial s}{\partial u} \right) \]

\[ = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i} \frac{\partial s}{\partial u} + \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \nabla_{\partial s} \left( \frac{\partial s}{\partial x_i} \right) \]

\[ = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i} \frac{\partial s}{\partial u} + \sum_{j=1}^{n} \frac{\partial x_j}{\partial u} \frac{\partial}{\partial x_j} \nabla_{\partial s} \left( \frac{\partial s}{\partial x_j} \right) \]

\[ = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i} \frac{\partial s}{\partial u} + \sum_{j=1}^{n} \frac{\partial x_j}{\partial u} \frac{\partial}{\partial x_j} \nabla_{\partial s} \left( \frac{\partial s}{\partial x_j} \right) \]
\[
= \sum_{i=1}^{n} \frac{\partial}{\partial u^i} \frac{\partial}{\partial v^i} + \sum_{i,j=1}^{n} \frac{\partial}{\partial u^i} \frac{\partial}{\partial v^j} \nabla_{\frac{\partial}{\partial u^i}} \left( \frac{\partial}{\partial v^j} \right)
\]

because \( \nabla_{\frac{\partial}{\partial u^i}} \left( \frac{\partial}{\partial v^j} \right) - \nabla_{\frac{\partial}{\partial v^j}} \left( \frac{\partial}{\partial u^i} \right) = [\frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^j}] = 0 \)

---

Two main results for geodesics:

M Riemannian:

The local result: (Prop. 3.6):

\( \forall t \in M, \ U \subset M \)

\( U \) a normal neighborhood of \( t \)

\( B \subset U \) a normal ball centered at \( t \).

\( \gamma : [0,1] \to B \) a geodesic

with \( \gamma(0) = t \)

Then, for all piecewise diff. curves,

\( c : [0,1] \to M \) s.t. \( c(0) = t \) and \( c(1) = \gamma(1) \),

\( l(c) \geq l(\gamma) \) with equality if \( c([0,1]) = \gamma([0,1]) \)
Def: \( p \in M \), \( U \subset M \) is called a normal neighborhood of \( p \) if
\[ U = \exp_p(V) \] where \( V \subset T_p M \) is an open set where \( \exp_p \) is a diffeomorphism.
A normal ball centered at \( p \) is the image of \( B_\varepsilon(0) \) for some \( \varepsilon > 0 \) s.t. \( \exp_p \) is a diffeo. on \( B_\varepsilon(0) \).
We usually denote
\[ B_\varepsilon(t) := \exp_p(B_\varepsilon(0)) \]
and \( \varepsilon \) is called the radius of \( B_\varepsilon(t) \).

The global result:
Suppose \( \gamma: [0, 1] \to M \) is a piecewise differentiable curve with constant speed (i.e., the parameter on \( \gamma \) is proportional to its length).
If the length of $\gamma$ is less than or equal to the length of any other piecewise differentiable curve from $\gamma(0)$ to $\gamma(1)$, then $\gamma$ is a geodesic. In particular, $\gamma$ is differentiable or regular.

Proof of the local result:

First assume $c([0, 1]) \subset B$.

$$l(c) = \int_0^1 |\frac{dc}{dt}| dt$$

Recall: $\exp_p(v) = \gamma(1, p, v)$.

$$\exp_p(tv) = \gamma(1, p, tv)$$

$$= \gamma(t, p, iv).$$

This shows that the radial line through 0 with direction $v$ maps to the geodesic with velocity $v$ at $t=0$. 
\[ \exp_p^{-1}(c(t)) = n(t)n(t), \]

where

\[ n : (0, 1) \to \mathbb{R} \]

\[ n : (0, 1) \to \mathbb{T}_{p} M. \]

and \( |n(t)| = 1 \quad \forall \ t \).

\( n, n \) piecewise differentiable.

Define \( f : (0, 1]^2 \to \mathbb{B} \)

\[ (n, t) \mapsto \exp_p(n n(t)) \]

Then \( c(t) = \exp_p(n(t)n(t)) \)

\[ dc \over dt = \frac{\partial f}{\partial n} n'(t) + \frac{\partial f}{\partial t} \]
Claim: \( \langle \frac{df}{dn}, \frac{df}{dt} \rangle = 0 \).

Admit this for the moment:

then \( \left| \frac{dc}{dt} \right|^2 = \left| \frac{df}{dn} \right|^2 (n'(t))^2 + \left| \frac{df}{dt} \right|^2 \).

\[
\frac{df}{dn} = (d(exp_p))_{n(t)} (n'(t))
\]

\[
\left| \frac{df}{dn} \right| = \left| d(exp_p)_{n(t)} (n'(t)) \right|
\]

\[
= |n'(t)| = 1
\]

\[
\frac{df}{dt} = (d(exp_p))_{n(t)} (n'(t))
\]

\[
e(c) = \lim_{\eta \to 0} \int_\eta^t \left| \frac{dc}{dt} \right| dt
\]

\[
\int_\eta^t \left| \frac{dc}{dt} \right| dt \geq \int_\eta^t \left| \frac{df}{dn} \right| n'(t) dt
\]

\[
= \int_\eta^t |n'(t)| n'(t) dt = \int_\eta^t n'(t) dt.
\]

\[
= n(1) - n(\eta) = e(t) - n(\eta)
\]

\( \eta \to 0 \) get \( e(c) \geq e(t) \).
If \( l(c) = l(\sigma) \), then \( \frac{df}{dt} = 0 \).

\[ \Rightarrow \quad v'(t) = 0, \quad \forall t. \]

\[ \Rightarrow \quad v \text{ is constant.} \]

\[ \Rightarrow \quad \text{exp}_{c}(0) \text{ is a line through the origin.} \]

\[ \Rightarrow \quad \text{is a geodesic} \]

\[ \Rightarrow \quad c = \sigma \quad \Rightarrow \quad c((0,1)) = \sigma((0,1)) \quad \square \]

**Proof of claim:** \( \left\langle \frac{\partial f}{\partial n}, \frac{\partial f}{\partial t} \right\rangle = 0. \)

\[ \frac{2}{\partial n} \left\langle \frac{\partial f}{\partial n}, \frac{\partial f}{\partial t} \right\rangle = \]

\[ \left\langle \frac{D}{\partial n} \left( \frac{\partial f}{\partial n} \right), \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial n}, \frac{D}{\partial t} \frac{\partial f}{\partial t} \right\rangle \]

\[ \frac{D}{\partial n} \left( \frac{\partial f}{\partial n} \right) = \text{covariant derivative of the velocity vector of geodesic} \]

\[ = 0. \]

Now use the symmetry lemma:

\[ \left\langle \frac{\partial f}{\partial n}, \frac{D}{\partial n} \frac{\partial f}{\partial t} \right\rangle = \left\langle \frac{\partial f}{\partial n}, \frac{D}{\partial t} \frac{\partial f}{\partial n} \right\rangle \]
\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial \nu}, \frac{\partial f}{\partial \nu} \right) \Rightarrow 0
\]

Length of velocity vector of a geodesic is constant.

\[
= \frac{1}{2} \frac{\partial}{\partial t} |v(t)| = 0
\]

So, \( \left< \frac{\partial f}{\partial \nu}, \frac{\partial f}{\partial t} \right> \) is independent of \( \nu \).

Plug in \( \nu = 0 \): we are at the origin of \( T_p M \) at \( t \).

\[
\frac{\partial f}{\partial \nu} = \nu, \quad \frac{\partial f}{\partial t} = 0
\]

(take the limit when \( \nu \to 0 \)) \( \Box \)