$$= \frac{1}{2} \frac{\partial}{\partial t} \left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \right\rangle \Rightarrow 0$$

~~length of velocity vector of a geodesic is constant.~~

$$= \frac{1}{2} \frac{\partial}{\partial t} |v(t)| = 0$$

So $\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle$ is independent of $r$.

plug in $r = 0$ : we are at the origin of $T_pM$ or at $p$.

$$\frac{\partial f}{\partial r} = v \qquad \frac{\partial f}{\partial t} = 0$$

(take ~~the~~ limit when $r \to 0$) $\blacksquare$

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If $c([0,1]) \not\subset B$.

by the continuity of $c$, $\exists$ a smallest $t > 0$ s.t. $c(t) \in S =$ boundary of $B$.

by the previous part

$$l(c) \geq l(c([0, t-\varepsilon])) \quad \forall \varepsilon > 0$$

$$\geq l(\gamma) \Rightarrow l(c) \geq l(\gamma) \quad \square$$

Def: $p \in M$, $W$ is a ~~normal~~ totally normal neighborhood of $p$ if $W$ is a normal neighborhood of all $q \in W$ and ~~$\forall p \in$~~ $\exists \delta > 0$ s.t. $\forall q \in W$, $W \subset B_\delta(q)$.

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Proof of the global result assuming the existence of ~~a~~ totally normal neighborhood.

$\gamma : [0,1] \longrightarrow M$.

$\forall t \in [0,1]$, $\exists$ totally normal neighborhood of $\gamma(t)$ where the local result holds, call this $W_t$.

$$\gamma([0,1]) \subset \bigcup_{t \in [0,1]} W_t.$$

compact.

$\exists\ t_1, \dots, t_n \in [0,1]$ s.t.

$$\gamma([0,1]) \subset \bigcup_{i=1}^{n} W_{t_i}$$

Consider $\gamma([0,1]) \cap W_i := W_{t_i}$ $i$ fixed

For any two points $s_1, s_2 \in [0,1]$
s.t. $\gamma([s_1, s_2]) \subset W_i$

$\exists!$ ~~unique~~ geodesic from $\gamma(s_1)$ to $\gamma(s_2)$ whose length is $\leq l(\gamma([s_1,s_2]))$

if $\gamma \neq$ geodesic, then $l(\gamma) >$ ~~the~~ length of geodesic and replacing this piece of $\gamma$ with the geodesic, we would get a shorter curve which would contradict the minimality of $l(\gamma)$. $\square$.