## 4.3 Sectional curvature

***Notation***. Here and throughout, let $M$ denote a Riemannian manifold of dimension $n \geq 2$, with metric $\langle \cdot, \cdot \rangle$, and $R\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{L}(\mathcal{X}(M))$ the curvature operator; for a given $p \in M$, we will abuse notation and write $R\colon T_pM \times T_pM \to \mathcal{L}(T_pM)$, which makes sense if we extend vectors $v \in T_pM$ to vector fields $V \in \mathcal{X}(M)$ in the usual way. Next, let $V$ be a real vector space (of dimension at least 2) equipped with an inner-product $\langle \cdot, \cdot \rangle$ (differentiating between the two uses of this symbol will be clear from context). Finally, for each $x, y \in V$ denote the area of the parallelogram determined by $x$ and $y$ by

$$|x \wedge y| := \sqrt{|x|^2|y|^2 - |\langle x, y \rangle|^2}.$$

This notation is not surprising if one is familiar with the *wedge product*; indeed, we may write $x \wedge y = |x \wedge y|(e_1 \wedge e_2)$ for some orthonormal basis $\{e_1, e_2\}$ for $\operatorname{span}\{x, y\}$. More generally, if $\dim(V) \geq m \geq 2$, then the wedge product gives $x_1 \wedge \cdots \wedge x_m = |x_1 \wedge \cdots \wedge x_m|(e_1 \wedge \cdots \wedge e_m)$ for some orthonormal basis $\{e_k\}_{k=1}^m$ for $\operatorname{span}\{x_k\}_{k=1}^m$, where $|x_1 \wedge \cdots \wedge x_m|$ denotes the volume of the parallelepiped determined by $\{x_k\}_{k=1}^m$.

***Definition***. Let $p \in M$ and $V \subset T_pM$ be a two-dimensional subspace. Define $K\colon V \times V \to \mathbb{R}$ by

$$K(x, y) := \begin{cases} \frac{\langle R(x,y)x, y \rangle}{|x \wedge y|^2} & \text{if } x, y \text{ are linearly independent,} \\ 0 & \text{otherwise.} \end{cases}$$

***Proposition (3.1)***. Let $p \in M$ and $V \subset T_pM$ be a two-dimensional subspace. If $(x, y), (u, v) \in V \times V$ are pairs of linearly independent vectors in $V$, then $K(x, y) = K(u, v)$.

*Proof.* First, observe that it is possible to transform the basis $\{x, y\}$ for $V$ into any other basis for $V$ using compositions of the operations

(a) $\{x, y\} \to \{y, x\}$;

(b) $\{x, y\} \to \{\lambda x, y\}$ for some nonzero $\lambda \in \mathbb{R}$;

(c) $\{x, y\} \to \{x + \lambda y, y\}$ for some $\lambda \in \mathbb{R}$.

Hence, it suffices to prove that $K$ is invariant under these operations. To this end, let $x, y \in V$ be linearly independent and note the following:

(a) Clearly, $|y \wedge x| = |x \wedge y|$, and so it suffices to show that $\langle R(y, x)y, x \rangle = \langle R(x, y)x, y \rangle$, which follows by applying part (b) of Proposition (2.5):

$$\langle R(y, x)y, x \rangle = -\langle R(x, y)y, x \rangle = \langle R(x, y)x, y \rangle.$$

(b) Suppose $\lambda \in \mathbb{R} \setminus \{0\}$. Since $|\lambda x \wedge y| = |\lambda||x \wedge y|$, it suffices to note that $\langle R(\lambda x, y)(\lambda x), y\rangle = \lambda^2 \langle R(x,y)x, y\rangle$ by the bilinearity of $R$ on $V \times V$ and linearity of $R(\cdot,\cdot)$ on $V$.

(c) Suppose $\lambda \in \mathbb{R}$. Then we have

$$\begin{aligned}
|(x+\lambda y) \wedge y|^2 &= |x+\lambda y|^2|y|^2 - |\langle x+\lambda y, y\rangle|^2 \\
&= \big(|x|^2 + 2\lambda\langle x,y\rangle + \lambda^2|y|^2\big)|y|^2 - \big(\langle x,y\rangle^2 + \lambda^2|y|^4 + 2\lambda\langle x,y\rangle|y|^2\big) \\
&= |x|^2|y|^2 - |\langle x,y\rangle|^2 \\
&= |x \wedge y|^2,
\end{aligned}$$

and so it remains to show that $\langle R(x+\lambda y, y)(x+\lambda y), y\rangle = \langle R(x,y)x, y\rangle$. For this, observe that the bilinearity of $R$ on $V \times V$ and linearity of $R(\cdot,\cdot)$ on $V$ yield

$$\begin{aligned}
\langle R(x+\lambda y, y)(x+\lambda y), y\rangle &= \langle R(x,y)(x+\lambda y), y\rangle + \lambda\langle R(y,y)(x+\lambda y), y\rangle \\
&= \langle R(x,y)x, y\rangle + \lambda\langle R(x,y)y, y\rangle \\
&\quad + \lambda\langle R(y,y)x, y\rangle + \lambda^2\langle R(y,y)y, y\rangle
\end{aligned}$$

and, hence, the result follows by applying parts (b) and (c) of Proposition (2.5) to obtain $\langle R(y,y), y, y\rangle = 0$ and $\langle R(y,y)x, y\rangle = \langle R(x,y)y, y\rangle = 0$. □

***Remark***. An immediate consequence of Proposition (3.1) is that the value of $K$ depends only the linear (in)dependence of its arguments and not on the arguments themselves. In particular, $K$ is constant over the set of pairs of linearly independent vectors in $V$; we denote this constant value by $K(V)$.

***Definition (3.2)***. Let $p \in M$ and $V \subset T_pM$ be a two-dimensional subspace. The *sectional curvature* of $V$ at $p$ is $K(V)$.

Sectional curvature is important because of its relationship to the curvature operator $R$. In particular, for any $p \in M$, knowing the values $\{K(V)\}_{V \subset T_pM}$ for all two-dimensional subspaces of $T_pM$ completely determines $R$. We make this precise with the following lemma:

***Lemma (3.3)***. Let $p \in M$ and $f\colon T_pM \times T_pM \times T_pM \to T_pM$ be a tri-linear mapping satisfying

(i) $\langle f(x,y,z), w\rangle + \langle f(y,z,x), w\rangle + \langle f(z,x,y), w\rangle = 0$;

(ii) $\langle f(x,y,z), w\rangle = -\langle f(y,x,z), w\rangle$;

(iii) $\langle f(x,y,z), w\rangle = -\langle f(x,y,w), z\rangle$;

(iv) $\langle f(x,y,z), w\rangle = \langle f(z,w,x), y\rangle$

for all $x, y, z, w \in T_pM$. For each two-dimensional subspace $V \subset T_pM$ define

$$\kappa(V) := \frac{\langle f(x,y,x), y\rangle}{|x \wedge y|^2}$$

for any pair $(x, y)$ of linearly independent vectors in $V$. If for each such $V$ we have $\kappa(V) = K(V)$, then $f(x, y, z) = R(x, y)z$ for all $x, y, z \in T_pM$.

***Remark***. Before we prove Lemma (3.3), it is necessary to remark that $\kappa$ is well-defined. Indeed, by the assumptions on $f$, verifying that $\kappa(V)$ is constant over the set of pairs of linearly independent vectors in $V$ is identical to the proof of Proposition (3.1).

*Proof of Lemma (3.3).* It suffices to prove $\langle f(x, y, z), w\rangle = \langle R(x, y)z, w\rangle$ for any $x, y, z, w \in T_pM$. To this end, observe that we have $\langle f(x, y, x), y\rangle = \langle R(x, y)x, y\rangle$ for all $x, y \in T_pM$; indeed, for linearly independent $x$ and $y$ this follows from the assumptions on $\kappa$, and for linearly dependent $x$ and $y$ it is easy to verify that $\langle f(x, y, x), y\rangle = 0 = \langle R(x, y)x, y\rangle$. In particular, for any $x, y, z \in T_pM$, we have

$$\langle f(x + z, y, x + z), y\rangle = \langle R(x + z, y)(x + z), y\rangle$$

and so by expanding we obtain

$$\begin{aligned}\langle f(x, y, x), y\rangle + 2\langle f(x, y, z), y\rangle + \langle f(z, y, z), y\rangle \\ = \langle R(x, y)x, y\rangle + 2\langle R(x, y)z, y\rangle + \langle R(z, y)z, y\rangle,\end{aligned}$$

which simplifies to $\langle f(x, y, z), y\rangle = \langle R(x, y)z, y\rangle$. Substituting $y + w$ for $y$ in this expression, expanding, and rearranging then yields

$$\langle f(x, y, z), w\rangle - \langle R(x, y)z, w\rangle = \langle f(y, z, x), w\rangle - \langle R(y, z)x, w\rangle,$$

from which it follows that $\langle f(x, y, z), w\rangle - \langle R(x, y)z, w\rangle$ is invariant under cyclic permutations of $(x, y, z)$. Applying assumption (i) and part (a) of Proposition (2.5), we therefore have

$$0 = 3\big(\langle f(x, y, z), w\rangle - \langle R(x, y)z, w\rangle\big);$$

the desired result is an immediate consequence of this equality. □

We now consider the case of constant sectional curvature; that is, for each $p \in M$, we require that $K(V) = K(W)$ for all two-dimensional subspaces $V, W \subset T_pM$. The following proposition (lemma in Do Carmo) provides a characterization of Riemannian manifolds with this property:

***Proposition (3.4)***. Let $p \in M$ and $g\colon T_pM \times T_pM \times T_pM \to T_pM$ be a tri-linear mapping satisfying

$$\langle g(x, y, z), w\rangle = \langle x, z\rangle\langle y, w\rangle - \langle y, z\rangle\langle x, w\rangle$$

for all $x, y, z, w \in T_pM$. Then the sectional curvature at $p$ is constant (and equal to $K_0 \in \mathbb{R}$) if and only if $K_0 g(x, y, z) = R(x, y)z$ for all $x, y, z \in T_pM$.

*Proof.* Before we begin, we observe some simple facts about $g$:

(i) Let $x, y, z, w \in T_pM$. Then

$$\begin{aligned}\langle g(x,y,z), w\rangle + \langle g(y,z,x), w\rangle + \langle g(z,x,y), w\rangle \\ = \langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle \\ + \langle y,x\rangle\langle z,w\rangle - \langle z,x\rangle\langle y,w\rangle \\ + \langle z,y\rangle\langle x,w\rangle - \langle x,y\rangle\langle z,w\rangle = 0.\end{aligned}$$

(ii) Let $x, y, z, w \in T_pM$. Then

$$\begin{aligned}\langle g(x,y,z), w\rangle &= \langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle \\ &= -\big(\langle y,z\rangle\langle x,w\rangle - \langle x,z\rangle\langle y,w\rangle\big) = -\langle g(y,x,z), w\rangle.\end{aligned}$$

(iii) Let $x, y, z, w \in T_pM$. Then

$$\begin{aligned}\langle g(x,y,z), w\rangle &= \langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle \\ &= -\big(\langle x,w\rangle\langle y,z\rangle - \langle y,w\rangle\langle x,z\rangle\big) = -\langle g(x,y,w), z\rangle.\end{aligned}$$

(iv) Let $x, y, z, w \in T_pM$. Then

$$\begin{aligned}\langle g(x,y,z), w\rangle &= \langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle \\ &= \langle z,x\rangle\langle w,y\rangle - \langle z,y\rangle\langle w,x\rangle = \langle g(z,w,x), y\rangle.\end{aligned}$$

With these facts in hand, we now prove each implication separately:
($\Rightarrow$) Assume that the sectional curvature at $p$ is constant (and equal to $K_0 \in \mathbb{R}$). Then by definition we have $\langle R(x,y)x, y\rangle = K_0|x \wedge y|^2$ for all $x, y \in T_pM$. Since we also have

$$\langle g(x,y,x), y\rangle = |x|^2|y|^2 - \langle x,y\rangle^2 = |x \wedge y|^2,$$

it follows that $K_0\langle g(x,y,x), y\rangle = \langle R(x,y)x, y\rangle$; that is

$$\frac{K_0\langle g(x,y,x), y\rangle}{|x \wedge y|^2} = K_0 = \frac{\langle R(x,y)x, y\rangle}{|x \wedge y|^2},$$

which is one of the assumptions of Lemma (3.3), provided we take $f := K_0 g$. As $g$ (and therefore $K_0 g$) satisfies properties (i)-(iv) above, the remaining assumptions of Lemma (3.3) are also satisfied (with $f = K_0 g$), and so applying the lemma yields $K_0 g(x,y,z) = R(x,y)z$ for all $x, y, z \in T_pM$, as desired.
($\Leftarrow$) Assume that $K_0 g(x,y,z) = R(x,y)z$ for all $x, y, z \in T_pM$ and some $K_0 \in \mathbb{R}$. Then, as we have already seen that $\langle g(x,y,x), y\rangle = |x \wedge y|^2$, for any two-dimensional subspace $V \subset T_pM$ and any pair $(x, y)$ of linearly independent vectors in $T_pM$ we have

$$K(V) = \frac{\langle R(x,y)x, y\rangle}{|x \wedge y|^2} = K_0$$

and, hence, the sectional curvature at $p$ is constant (and equal to $K_0$). $\square$

***Corollary (3.5)***. Let $p \in M$ and $\{e_k\}_{k=1}^n$ be an orthonormal basis for $T_pM$. For each $i, j, k, \ell \in \{1, \ldots, n\}$, define $R_{ijk\ell} := \langle R(e_i, e_j)e_k, e_\ell \rangle$. Then the sectional curvature at $p$ is constant (and equal to $K_0 \in \mathbb{R}$) if and only if $R_{ijk\ell} = K_0(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell})$ for all $i, j, k, \ell \in \{1, \ldots, n\}$, where $\delta_{ab}$ denotes the Kronecker delta.

*Proof.* By Proposition (3.4), the sectional curvature at $p$ is constant (and equal to $K_0$) if and only if $K_0 g(x, y, z) = R(x, y)z$ for all $x, y, z, w \in T_pM$, where $g\colon T_pM \times T_pM \times T_pM \to T_pM$ is a tri-linear mapping satisfying $\langle g(x, y, z), w \rangle = \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle$ for all $x, y, z, w \in T_pM$. Hence, the sectional curvature is constant if and only if

$$\begin{aligned} R_{ijk\ell} &= \langle R(e_i, e_j)e_k, e_\ell \rangle \\ &= K_0 \langle g(e_i, e_j, e_k), e_\ell \rangle = K_0\big(\langle e_i, e_k \rangle \langle e_j, e_\ell \rangle - \langle e_j, e_k \rangle \langle e_i, e_\ell \rangle\big) = K_0(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}), \end{aligned}$$

which is the desired result. □

***Remark***. The condition $R_{ijk\ell} = K_0(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})$ for all $i, j, k, \ell \in \{1, \ldots, n\}$ is equivalent to having, for each pair $\{i, j\} \subset \{1, \ldots, n\}$, $R_{ijij} = -R_{ijji} = K_0(1 - \delta_{ij})$ and $R_{ijk\ell} = 0$ for all $k, \ell \in \{1, \ldots, n\} \setminus \{i, j\}$.

***Examples***. Suppose the sectional curvature at $p \in M$ is constant, with value $K_0 \in \mathbb{R}$. Then, modulo scaling the metric on $M$, there are only three unique cases to consider:

(1) If $K_0 = 0$, then $M$ is locally isometric to $\mathbb{R}^n$, $n$-dimensional Euclidean space;

(2) If $K_0 = 1$, then $M$ is locally isometric to $\mathbb{S}^n$, the unit $n$-sphere;

(3) If $K_0 = -1$, then $M$ is locally isometric to $\mathbb{H}^n$, $n$-dimensional hyperbolic space.

(See Figure 1 below for an illustration when $n = 2$.) Riemannian manifolds of constant sectional curvature are sometimes referred to as *space forms*.

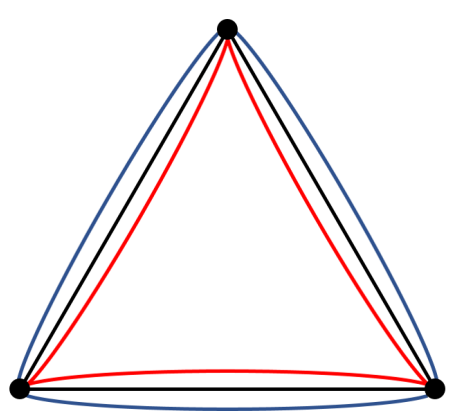

**Figure 1**. *A geodesic triangle*. Three points around $p \in M$ are connected via geodesic curves and projected onto $\mathbb{R}^2$. In black, $M \cong \mathbb{R}^2$ ($K_0 = 0$); in blue, $M \cong \mathbb{S}^2$ ($K_0 = 1$); in red, $M \cong \mathbb{H}^2$ ($K_0 = -1$).

## 4.4 Ricci and scalar curvature

The following is a brief introduction to Ricci and scalar curvature. Only basic definitions and results are provided, and all proofs are omitted. The notation of section 4.3 is assumed throughout.

***Notation***. Let $p \in M$ and $x_0 \in T_pM$ such that $|x_0| = 1$. Denote by $\{x_k\}_{k=1}^{n-1} \subset T_pM$ an orthonormal basis for the hyperplane in $T_pM$ which is orthogonal to $\mathrm{span}\{x_0\}$.

***Definition***. The *Ricci curvature* at $p$ in the direction $x_0$ is defined by

$$\mathrm{Ric}_p(x_0) := \frac{1}{n-1}\sum_{k=1}^{n-1}\langle R(x_0, x_k)x_0, x_k\rangle.$$

Moreover, the *scalar curvature* at $p$ is defined by

$$S(p) := \frac{1}{n}\sum_{j=0}^{n-1}\mathrm{Ric}_p(x_j) = \frac{1}{n(n-1)}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\langle R(x_j, x_k)x_j, x_k\rangle$$

***Proposition***. The Ricci curvature at $p$ in the direction $x_0$ is independent of the choice of orthonormal basis for the hyperplane in $T_pM$ which is orthogonal to $\mathrm{span}\{x_0\}$. That is, if $\{x_k\}_{k=1}^{n-1}$ and $\{z_k\}_{k=1}^{n-1}$ are two such orthonormal bases, then

$$\sum_{k=1}^{n-1}\langle R(x_0, x_k)x_0, x_k\rangle = \sum_{k=1}^{n-1}\langle R(x_0, z_k)x_0, z_k\rangle.$$

***Corollary***. The scalar curvature at $p$ is independent of the choice of orthonormal basis for $T_pM$.