4.3 Sectional curvature

**Notation.** Here and throughout, let $M$ denote a Riemannian manifold of dimension $n \geq 2$, with metric $\langle \cdot, \cdot \rangle$, and $R : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{L}(\mathcal{X}(M))$ the curvature operator; for a given $p \in M$, we will abuse notation and write $R : T_p M \times T_p M \to \mathcal{L}(T_p M)$, which makes sense if we extend vectors $v \in T_p M$ to vector fields $V \in \mathcal{X}(M)$ in the usual way. Next, let $V$ be a real vector space (of dimension at least 2) equipped with an inner-product $\langle \cdot, \cdot \rangle$ (differentiating between the two uses of this symbol will be clear from context). Finally, for each $x, y \in V$ denote the area of the parallelogram determined by $x$ and $y$ by

$$|x \wedge y| := \sqrt{|x|^2 |y|^2 - |\langle x, y \rangle|^2}.$$ 

This notation is not surprising if one is familiar with the wedge product; indeed, we may write $x \wedge y = |x \wedge y| (e_1 \wedge e_2)$ for some orthonormal basis $\{e_1, e_2\}$ for $\text{span}\{x, y\}$. More generally, if $	ext{dim}(V) \geq m \geq 2$, then the wedge product gives $x_1 \wedge \cdots \wedge x_m = |x_1 \wedge \cdots \wedge x_m| (e_1 \wedge \cdots \wedge e_m)$ for some orthonormal basis $\{e_k\}_{k=1}^m$ for $\text{span}\{x_k\}_{k=1}^m$, where $|x_1 \wedge \cdots \wedge x_m|$ denotes the volume of the parallelepiped determined by $\{x_k\}_{k=1}^m$.

**Definition.** Let $p \in M$ and $V \subset T_p M$ be a two-dimensional subspace. Define $K : V \times V \to \mathbb{R}$ by

$$K(x, y) := \begin{cases} \frac{\langle R(x,y)x, y \rangle}{|x \wedge y|^2} & \text{if } x, y \text{ are linearly independent}, \\ 0 & \text{otherwise}. \end{cases}$$

**Proposition (3.1).** Let $p \in M$ and $V \subset T_p M$ be a two-dimensional subspace. If $(x, y), (u, v) \in V \times V$ are pairs of linearly independent vectors in $V$, then $K(x, y) = K(u, v)$.

**Proof.** First, observe that it is possible to transform the basis $\{x, y\}$ for $V$ into any other basis for $V$ using compositions of the operations

(a) $\{x, y\} \to \{y, x\}$;

(b) $\{x, y\} \to \{\lambda x, y\}$ for some nonzero $\lambda \in \mathbb{R}$;

(c) $\{x, y\} \to \{x + \lambda y, y\}$ for some $\lambda \in \mathbb{R}$.

Hence, it suffices to prove that $K$ is invariant under these operations. To this end, let $x, y \in V$ be linearly independent and note the following:

(a) Clearly, $|y \wedge x| = |x \wedge y|$, and so it suffices to show that $\langle R(y, x)y, x \rangle = \langle R(x, y)x, y \rangle$, which follows by applying part (b) of Proposition (2.5):

$$\langle R(y, x)y, x \rangle = -\langle R(x, y)y, x \rangle = \langle R(x, y)x, y \rangle.$$

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(b) Suppose \( \lambda \in \mathbb{R} \setminus \{0\} \). Since \( |\lambda x \wedge y| = |\lambda||x \wedge y| \), it suffices to note that \( \langle R(\lambda x, y)(\lambda x), y, z \rangle \) by the bilinearity of \( R \) on \( V \times V \) and linearity of \( R(\cdot, \cdot) \) on \( V \).

(c) Suppose \( \lambda \in \mathbb{R} \). Then we have

\[
|(x + \lambda y) \wedge y|^2 = |x + \lambda y|^2 - |x + \lambda y, y|^2
\]

\[
= (|x|^2 + 2\lambda \langle x, y \rangle + \lambda^2 |y|^2) |y|^2 - (\langle x, y \rangle^2 + \lambda^2 |y|^4 + 2\lambda \langle x, y \rangle |y|^2)
\]

\[
= |x|^2 |y|^2 - |\langle x, y \rangle|^2
\]

and so it remains to show that \( \langle R(x + \lambda y, y)(x + \lambda y), y \rangle = \langle R(x, y)x, y \rangle \). For this, observe that the bilinearity of \( R \) on \( V \times V \) and linearity of \( R(\cdot, \cdot) \) on \( V \) yield

\[
\langle R(x + \lambda y, y)(x + \lambda y), y \rangle = \langle R(x, y)(x + \lambda y), y \rangle + \lambda \langle R(y, y)(x + \lambda y), y \rangle
\]

\[
= \langle R(x, y)x, y \rangle + \lambda \langle R(y, y)x, y \rangle + \lambda^2 \langle R(y, y)y, y \rangle
\]

and, hence, the result follows by applying parts (b) and (c) of Proposition (2.5) to obtain \( \langle R(y, y), y, y \rangle = 0 \) and \( \langle R(y, y)x, y \rangle = \langle R(x, y)y, y \rangle = 0 \).

\[\square\]

**Remark.** An immediate consequence of Proposition (3.1) is that the value of \( K \) depends only the linear (in)dependence of its arguments and not on the arguments themselves. In particular, \( K \) is constant over the set of pairs of linearly independent vectors in \( V \); we denote this constant value by \( K(V) \).

**Definition (3.2).** Let \( p \in M \) and \( V \subset T_p M \) be a two-dimensional subspace. The sectional curvature of \( V \) at \( p \) is \( K(V) \).

Sectional curvature is important because of its relationship to the curvature operator \( R \). In particular, for any \( p \in M \), knowing the values \( \{K(V)\}_{V \subset T_p M} \) for all two-dimensional subspaces of \( T_p M \) completely determines \( R \). We make this precise with the following lemma:

**Lemma (3.3).** Let \( p \in M \) and \( f: T_p M \times T_p M \times T_p M \rightarrow T_p M \) be a tri-linear mapping satisfying

(i) \( \langle f(x, y, z), w \rangle + \langle f(y, z, x), w \rangle + \langle f(z, x, y), w \rangle = 0 \);

(ii) \( \langle f(x, y, z), w \rangle = -\langle f(y, x, z), w \rangle \);

(iii) \( \langle f(x, y, z), w \rangle = -\langle f(x, y, w), z \rangle \);

(iv) \( \langle f(x, y, z), w \rangle = \langle f(z, w, x), y \rangle \)

for all \( x, y, z, w \in T_p M \). For each two-dimensional subspace \( V \subset T_p M \) define

\[
\kappa(V) := \frac{\langle f(x, y, x), y \rangle}{|x \wedge y|^2}
\]
for any pair \((x, y)\) of linearly independent vectors in \(V\). If for each such \(V\) we have \(\kappa(V) = K(V)\), then \(f(x, y, z) = R(x, y)z\) for all \(x, y, z \in T_pM\).

**Remark.** Before we prove Lemma (3.3), it is necessary to remark that \(\kappa\) is well-defined. Indeed, by the assumptions on \(f\), verifying that \(\kappa\) is constant over the set of pairs of linearly independent vectors in \(V\) is identical to the proof of Proposition (3.1).

**Proof of Lemma (3.3).** It suffices to prove \(\langle f(x, y, z), w \rangle = \langle R(x, y)z, w \rangle\) for any \(x, y, z, w \in T_pM\). To this end, observe that we have \(\langle f(x, y, y), y \rangle = \langle R(x, y)x, y \rangle\) for all \(x, y \in T_pM\); indeed, for linearly independent \(x\) and \(y\) this follows from the assumptions on \(\kappa\), and for linearly dependent \(x\) and \(y\) it is easy to verify that \(\langle f(x, y, x), y \rangle = 0 = \langle R(x, y)x, y \rangle\). In particular, for any \(x, y, z \in T_pM\), we have \(\langle f(x + z, y, x + z), y \rangle = \langle R(x + z, y)(x + z), y \rangle\) and so by expanding we obtain
\[
\langle f(x, y, x), y \rangle + 2\langle f(x, y, z), y \rangle + \langle f(z, y, z), y \rangle = \langle R(x, y)x, y \rangle + 2\langle R(x, y)z, y \rangle + \langle R(z, y)z, y \rangle,
\]
which simplifies to \(\langle f(x, y, z), y \rangle = \langle R(x, y)z, y \rangle\). Substituting \(y + w\) for \(y\) in this expression, expanding, and rearranging then yields
\[
\langle f(x, y, z), w \rangle - \langle R(x, y)z, w \rangle = \langle f(y, z, x), w \rangle - \langle R(y, z)x, w \rangle,
\]
from which it follows that \(\langle f(x, y, z), w \rangle - \langle R(x, y)z, w \rangle\) is invariant under cyclic permutations of \((x, y, z)\). Applying assumption (i) and part (a) of Proposition (2.5), we therefore have
\[
0 = 3\left(\langle f(x, y, z), w \rangle - \langle R(x, y)z, w \rangle\right);
\]
the desired result is an immediate consequence of this equality.

We now consider the case of constant sectional curvature; that is, for each \(p \in M\), we require that \(K(V) = K(W)\) for all two-dimensional subspaces \(V, W \subset T_pM\). The following proposition (lemma in Do Carmo) provides a characterization of Riemannian manifolds with this property:

**Proposition (3.4).** Let \(p \in M\) and \(g : T_pM \times T_pM \times T_pM \to T_pM\) be a tri-linear mapping satisfying
\[
\langle g(x, y, z), w \rangle = \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle
\]
for all \(x, y, z, w \in T_pM\). Then the sectional curvature at \(p\) is constant (and equal to \(K_0 \in \mathbb{R}\)) if and only if \(K_0 g(x, y, z) = R(x, y)z\) for all \(x, y, z \in T_pM\).

**Proof.** Before we begin, we observe some simple facts about \(g\):

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Let $x, y, z, w \in T_pM$. Then

\[
\langle g(x, y, z), w \rangle + \langle g(y, z, x), w \rangle + \langle g(z, x, y), w \rangle \\
= \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle \\
+ \langle y, x \rangle \langle z, w \rangle - \langle z, x \rangle \langle y, w \rangle \\
+ \langle z, y \rangle \langle x, w \rangle - \langle x, y \rangle \langle z, w \rangle = 0.
\]

(ii) Let $x, y, z, w \in T_pM$. Then

\[
\langle g(x, y, z), w \rangle = \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle = -\left(\langle y, z \rangle \langle x, w \rangle - \langle x, z \rangle \langle y, w \rangle \right) = -\langle g(y, x, z), w \rangle.
\]

(iii) Let $x, y, z, w \in T_pM$. Then

\[
\langle g(x, y, z), w \rangle = \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle = -\left(\langle x, w \rangle \langle y, z \rangle - \langle y, w \rangle \langle x, z \rangle \right) = -\langle g(x, y, w), z \rangle.
\]

(iv) Let $x, y, z, w \in T_pM$. Then

\[
\langle g(x, y, z), w \rangle = \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle \\
= \langle z, x \rangle \langle w, y \rangle - \langle z, y \rangle \langle w, x \rangle = \langle g(z, w, x), y \rangle.
\]

With these facts in hand, we now prove each implication separately:

$(\Rightarrow)$ Assume that the sectional curvature at $p$ is constant (and equal to $K_0 \in \mathbb{R}$). Then by definition we have $\langle R(x, y)x, y \rangle = K_0|x \wedge y|^2$ for all $x, y \in T_pM$. Since we also have

\[
\langle g(x, y, x), y \rangle = |x|^2|y|^2 - \langle x, y \rangle^2 = |x \wedge y|^2,
\]

it follows that $K_0\langle g(x, y, x), y \rangle = \langle R(x, y)x, y \rangle$; that is

\[
\frac{K_0\langle g(x, y, x), y \rangle}{|x \wedge y|^2} = K_0 = \frac{\langle R(x, y)x, y \rangle}{|x \wedge y|^2},
\]

which is one of the assumptions of Lemma (3.3), provided we take $f := K_0g$. As $g$ (and therefore $K_0g$) satisfies properties (i)-(iv) above, the remaining assumptions of Lemma (3.3) are also satisfied (with $f = K_0g$), and so applying the lemma yields $K_0g(x, y, z) = R(x, y)z$ for all $x, y, z \in T_pM$, as desired.

$(\Leftarrow)$ Assume that $K_0g(x, y, z) = R(x, y)z$ for all $x, y, z \in T_pM$ and some $K_0 \in \mathbb{R}$. Then, as we have already seen that $\langle g(x, y, x), y \rangle = |x \wedge y|^2$, for any two-dimensional subspace $V \subset T_pM$ and any pair $(x, y)$ of linearly independent vectors in $T_pM$ we have

\[
K(V) = \frac{\langle R(x, y)x, y \rangle}{|x \wedge y|^2} = K_0
\]

and, hence, the sectional curvature at $p$ is constant (and equal to $K_0$). \qed
Corollary (3.5). Let \( p \in M \) and \( \{e_k\}_{k=1}^{n} \) be an orthonormal basis for \( T_pM \). For each \( i, j, k, \ell \in \{1, \ldots, n\} \), define \( R_{ijk\ell} := \langle R(e_i, e_j)e_k, e_\ell \rangle \). Then the sectional curvature at \( p \) is constant (and equal to \( K_0 \in \mathbb{R} \)) if and only if \( R_{ijk\ell} = K_0(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}) \) for all \( i, j, k, \ell \in \{1, \ldots, n\} \), where \( \delta_{ab} \) denotes the Kronecker delta.

Proof. By Proposition (3.4), the sectional curvature at \( p \) is constant (and equal to \( K_0 \)) if and only if
\[
K_0 g(x, y, z) = R(x, y)z \quad \text{for all} \quad x, y, z, w \in T_pM,
\]
where \( g : T_pM \times T_pM \times T_pM \to T_pM \) is a tri-linear mapping satisfying
\[
\langle g(x, y, z), w \rangle = \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle \quad \text{for all} \quad x, y, z, w \in T_pM.
\]
Hence, the sectional curvature is constant if and only if
\[
R_{ijk\ell} = \langle R(e_i, e_j)e_k, e_\ell \rangle = K_0 \left( \langle e_i, e_k \rangle \langle e_j, e_\ell \rangle - \langle e_j, e_k \rangle \langle e_i, e_\ell \rangle \right) = K_0(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}),
\]
which is the desired result. \( \square \)

Remark. The condition \( R_{ijk\ell} = K_0(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}) \) for all \( i, j, k, \ell \in \{1, \ldots, n\} \) is equivalent to having, for each pair \( \{i, j\} \subseteq \{1, \ldots, n\} \), \( R_{iijj} = -R_{ijji} = K_0(1 - \delta_{ij}) \) and \( R_{ijk\ell} = 0 \) for all \( k, \ell \in \{1, \ldots, n\} \setminus \{i, j\} \).

Examples. Suppose the sectional curvature at \( p \in M \) is constant, with value \( K_0 \in \mathbb{R} \). Then, modulo scaling the metric on \( M \), there are only three unique cases to consider:

(1) If \( K_0 = 0 \), then \( M \) is locally isometric to \( \mathbb{R}^n \), \( n \)-dimensional Euclidean space;

(2) If \( K_0 = 1 \), then \( M \) is locally isometric to \( S^n \), the unit \( n \)-sphere;

(3) If \( K_0 = -1 \), then \( M \) is locally isometric to \( \mathbb{H}^n \), \( n \)-dimensional hyperbolic space.

(See Figure 1 below for an illustration when \( n = 2 \).) Riemannian manifolds of constant sectional curvature are sometimes referred to as space forms.

Figure 1. A geodesic triangle. Three points around \( p \in M \) are connected via geodesic curves and projected onto \( \mathbb{R}^2 \). In black, \( M \cong \mathbb{R}^2 \) (\( K_0 = 0 \)); in blue, \( M \cong S^2 \) (\( K_0 = 1 \)); in red, \( M \cong \mathbb{H}^2 \) (\( K_0 = -1 \)).
4.4 Ricci and scalar curvature

The following is a brief introduction to Ricci and scalar curvature. Only basic definitions and results are provided, and all proofs are omitted. The notation of section 4.3 is assumed throughout.

**Notation.** Let $p \in M$ and $x_0 \in T_p M$ such that $|x_0| = 1$. Denote by $\{x_k\}_{k=1}^{n-1} \subset T_p M$ an orthonormal basis for the hyperplane in $T_p M$ which is orthogonal to $\text{span}\{x_0\}$.

**Definition.** The Ricci curvature at $p$ in the direction $x_0$ is defined by

$$\text{Ric}_p(x_0) := \frac{1}{n-1} \sum_{k=1}^{n-1} \langle R(x_0, x_k)x_0, x_k \rangle.$$

Moreover, the scalar curvature at $p$ is defined by

$$S(p) := \frac{1}{n} \sum_{j=0}^{n-1} \text{Ric}_p(x_j) = \frac{1}{n(n-1)} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \langle R(x_j, x_k)x_j, x_k \rangle.$$

**Proposition.** The Ricci curvature at $p$ in the direction $x_0$ is independent of the choice of orthonormal basis for the hyperplane in $T_p M$ which is orthogonal to $\text{span}\{x_0\}$. That is, if $\{x_k\}_{k=1}^{n-1}$ and $\{z_k\}_{k=1}^{n-1}$ are two such orthonormal bases, then

$$\sum_{k=1}^{n-1} \langle R(x_0, x_k)x_0, x_k \rangle = \sum_{k=1}^{n-1} \langle R(x_0, z_k)x_0, z_k \rangle.$$

**Corollary.** The scalar curvature at $p$ is independent of the choice of orthonormal basis for $T_p M$. 
