Differentiable manifolds:

Generalize regular surfaces in $\mathbb{R}^3$.

Recall: A regular surface in $\mathbb{R}^3$ is a subset $M \subset \mathbb{R}^3$ s.t. $\forall\, p \in M$, $\exists$ neighborhood V of p in $\mathbb{R}^3$ and a mapping $\varphi : U \to V$ where U is an open subset of $\mathbb{R}^2$ s.t.

(a) φ is differentiable and φ is a homeomorphism from U to $V \cap M$

(b) the differential $(d\varphi)_q : T_q U \to T_{\varphi(q)} V$ is injective $\forall\, q \in U$.

$$
\begin{array}{ccc}
T_q U & \longrightarrow & T_{\varphi(q)} V \\
\cong & & \cong \\
T_q \mathbb{R}^2 & & T_{\varphi(q)} \mathbb{R}^3 \\
\cong & & \cong \\
\mathbb{R}^2 & & \mathbb{R}^3
\end{array}
$$

Def: The U as above are called coordinate charts of M.

Note: $\forall$ two charts U_1, U_2 $\quad \varphi_1 : U_1 \longrightarrow M$

$\varphi_2 : U_2 \longrightarrow M$

s.t. $\varphi_1(U_1) \cap \varphi_2(U_2) \neq \emptyset$

put $\quad W := \varphi_1(U_1) \cap \varphi_2(U_2)$

$$\begin{array}{l} \varphi_2^{-1} \circ \varphi_1 : \varphi_1^{-1}(W) \xrightarrow{\varphi_1} W \xrightarrow{\varphi_2^{-1}} U_2 \subset \mathbb{R}^2 \\ \qquad\qquad\quad \cap \\ \qquad\qquad\quad U_1 \subset \mathbb{R}^2 \end{array}$$

$\varphi_2^{-1} \circ \varphi_1$ is differentiable.

$\varphi_1 : U_1 \longrightarrow M \cap V_1 \qquad (d\varphi_1)_q : T_q U_1 \xrightarrow{\cong} T_{\varphi_1(q)}(M \cap V_1)$

Abstract manifolds!

Definition: A differentiable manifold of dim. n is a set M together with the data of a collection of open subsets $\{V_\alpha\}$ of $\mathbb{R}^n$ and injective maps

$$\varphi_\alpha : V_\alpha \longrightarrow M \quad \text{s.t.}$$

(1) $$\bigcup \varphi_\alpha(V_\alpha) = M$$

(2) $\forall\ \alpha, \beta$ s.t. $$\underbrace{\varphi_\alpha(V_\alpha) \cap \varphi_\beta(V_\beta)}_{\substack{\shortparallel \\ W_{\alpha\beta}}} \neq \phi$$

$$\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(W_{\alpha\beta}) \longrightarrow \mathbb{R}^n$$

$$\cup$$

$$\left(\longrightarrow \varphi_\beta^{-1}(W_{\alpha\beta}) \subset V_\beta \right)$$

is differentiable.

Remark: The differentiable structure induces a topology on M:

A subset W of M is open $\iff$

$\forall \alpha$, $\varphi_\alpha^{-1}(W \cap \varphi(U_\alpha))$ is open.

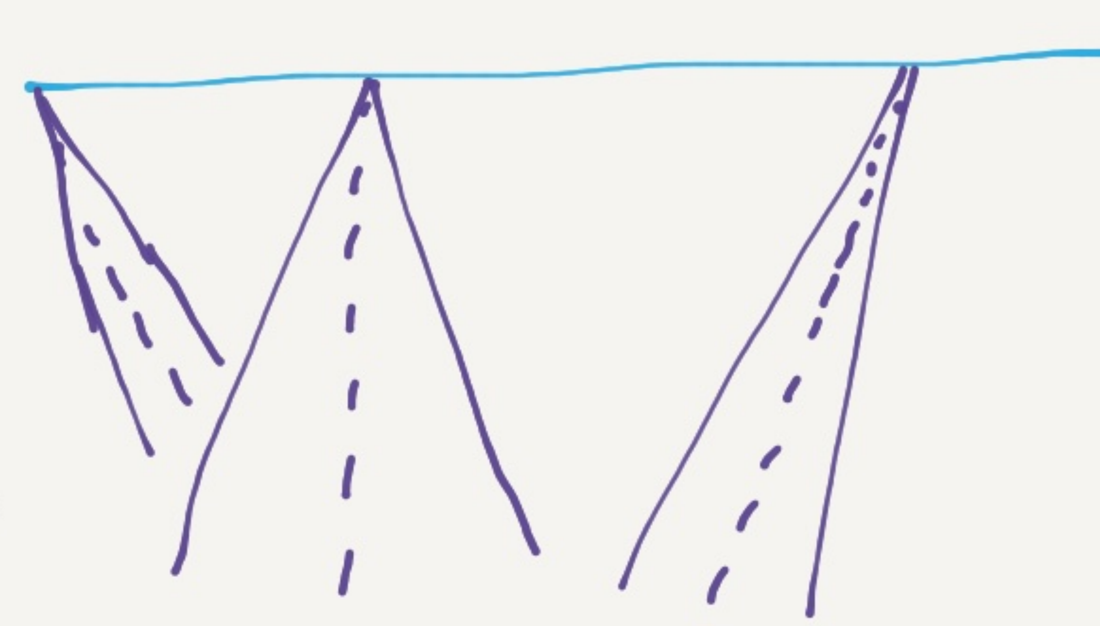
In this topology, $\varphi_\alpha(U_\alpha)$ is open $\forall \alpha$ and φ_α is a homeomorphism to its image $\forall \alpha$ and $\varphi_\alpha : U_\alpha \to M$ is continuous.

Example: **Real projective space:** $\mathbb{P}^n(\mathbb{R})$

$$\mathbb{P}^0(\mathbb{R}) = \{pt\} \qquad \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}.$$

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \text{circle}/\{\pm 1\} = \mathbb{R}^2 \cup \text{all directions}$$

$$= \mathbb{R}^2 \cup \text{points on the horizon}$$



different roads converge to different points on the horizon: parallel lines meet at infinity.