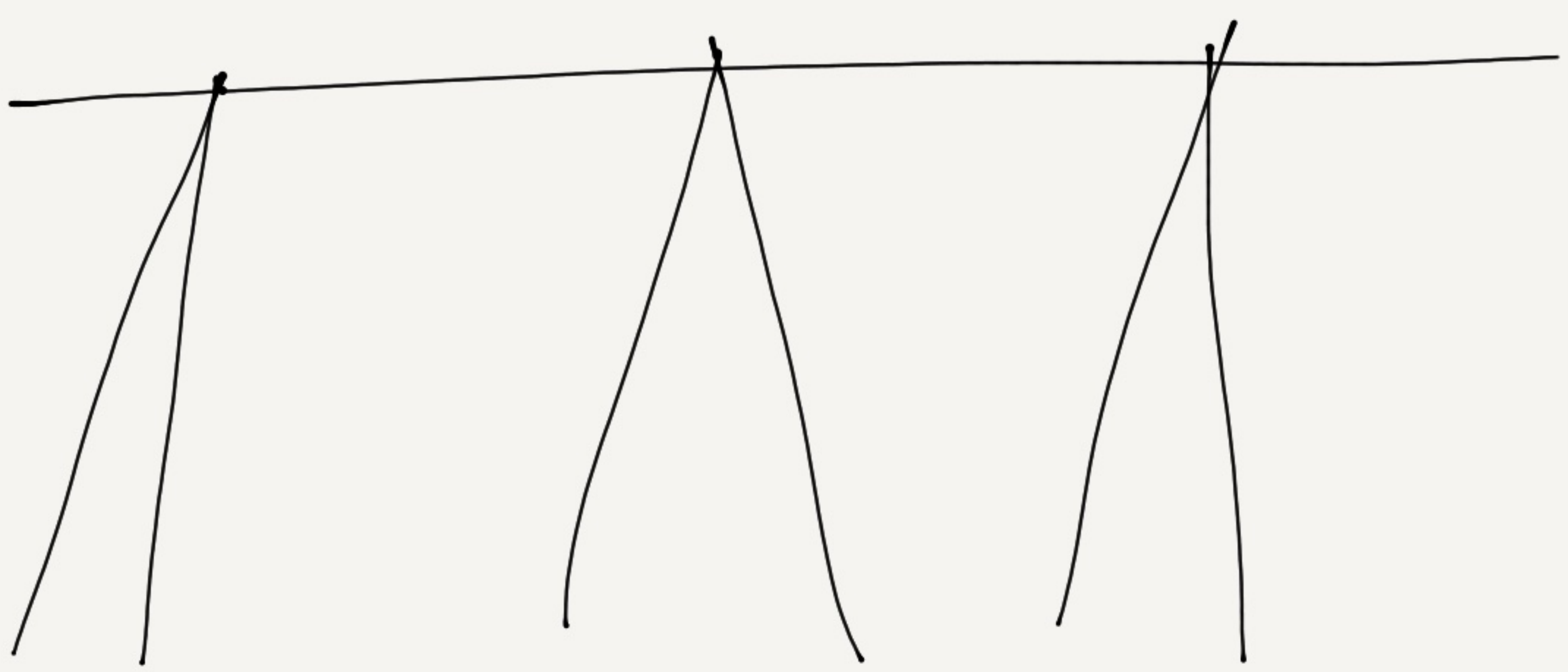
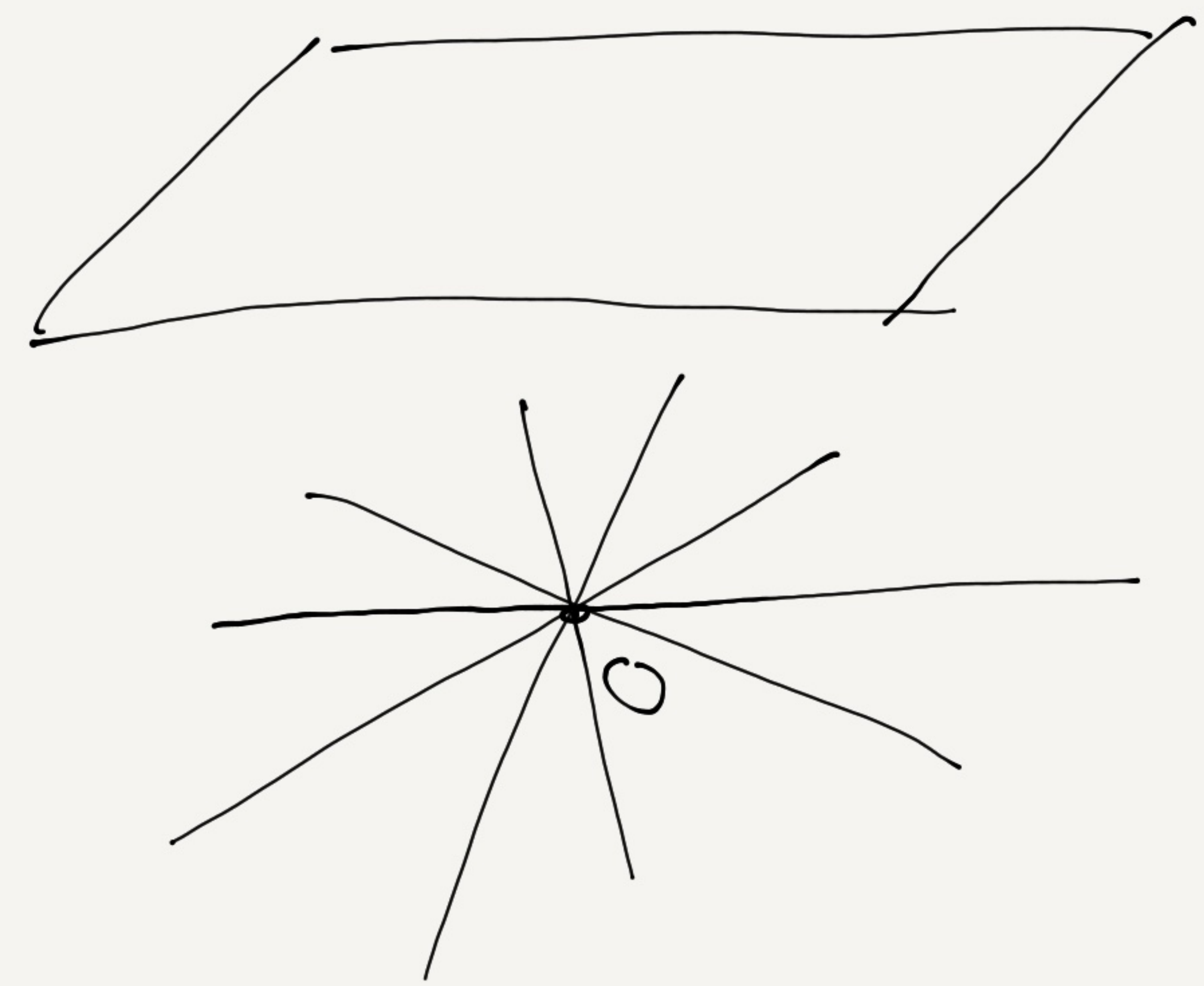
$$\mathbb{P}^2(\mathbb{R}) = \underset{\substack{\uparrow \\ \text{Euclidean} \\ \text{plane}}}{\mathbb{R}^2} \cup \underset{\substack{\uparrow \\ \text{horizon}}}{\mathbb{R}} \cup \underset{\substack{\uparrow \\ \text{the horizontal direction.}}}{\{\infty\}}$$

$$= \mathbb{R}^2 \cup \text{all directions in } \mathbb{R}^2$$

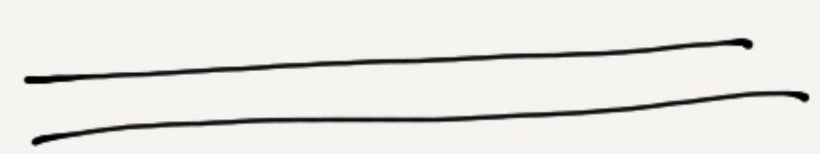
$$\cup \text{ all lines through } 0 \text{ in } \mathbb{R}^2$$



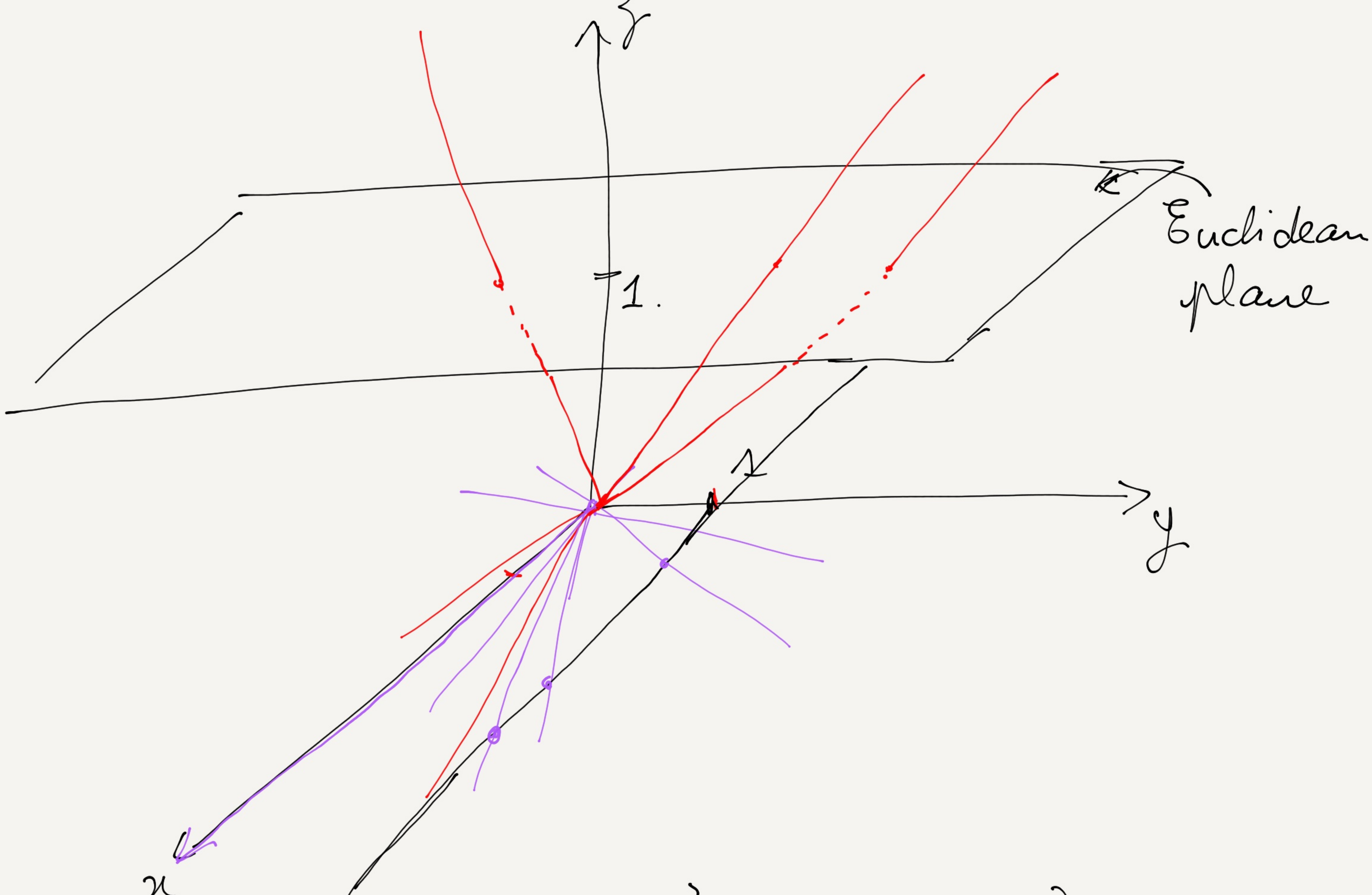
$\rightsquigarrow$



$+ \{\infty\} =$



$\rightsquigarrow$



$\rightsquigarrow \mathbb{P}^2(\mathbb{R}) = \{\text{lines in } \mathbb{R}^3 \text{ through } 0\}.$

$\mathbb{P}^n(\mathbb{R}) = \{\text{ " " } \mathbb{R}^{n+1} \text{ " " }\}.$

Homogeneous coordinates:

line through 0 $\Leftarrow$ point $\neq 0$

$(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ give the same line through 0 when $\exists \lambda \neq 0$, $\lambda \in \mathbb{R}$ s.t.

$$(y_0, \ldots, y_n) = \lambda (x_0, \ldots, x_n)$$

point of $\mathbb{P}^n(\mathbb{R}) \Longleftrightarrow (x_0, \ldots, x_n)$ well-defined up to multiplication by a nonzero scalar

We call $(x_0, \ldots, x_n)$ homogeneous coordinates.

$\forall$ any point of $\mathbb{P}^n$ with coordinates $(x_0, \ldots, x_n)$

$\exists\, i$ s.t. $x_i \neq 0$.

Manifold structure: $V_i := \{(x_0, \ldots, x_n) : x_i \neq 0\} \subset \mathbb{P}^n$

If $p \in V_i$, then if $(x_0, \dots, x_n)$ is a coordinate for p, so is $\frac{1}{x_i}(x_0, \dots, x_n)$ is also a coordinate for p.

$$\frac{1}{x_i}(x_0, \dots, x_n) = (x_0, \dots, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, \dots, x_n)$$

$$\varphi_i : U_i = \mathbb{R}^n \longrightarrow V_i \subset \mathbb{P}^n(\mathbb{R})$$

$$(a_1, \dots, a_n) \longmapsto (a_1, \dots, \underset{\llcorner i\text{-th}}{1}, \dots, a_n)$$

affine coordinates.

$$\mathbb{P}^n = \bigcup_{i=0}^{n} V_i$$

Compatibility or coordinate changes have to be differentiable $\forall\ i, j$. For simplicity: $i=0$, $j=1$

$$\varphi_0 : U_0 \longrightarrow V_0 \subset \mathbb{P}^n(\mathbb{R})$$

$$(a_1, \dots, a_n) \longmapsto (1, a_1, \dots, a_n)$$

$$\varphi_1 : U_1 \longrightarrow V_1 \subset \mathbb{P}^n(\mathbb{R})$$

$$(b_1, \dots, b_n) \longmapsto (b_1, 1, b_2, \dots, b_n)$$

$$W_{01} := \underset{\substack{\| \\ V_0}}{\varphi_0(U_0)} \cap \underset{\substack{\| \\ V_1}}{\varphi_1(U_1)} = \{(x_0, \dots, x_n) : x_0 \neq 0,\ x_1 \neq 0\}$$

$$\begin{tikzcd}
\varphi_1^{-1} \circ \varphi_0 : & \varphi_0^{-1}(W_{01}) \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & \varphi_1^{-1}(W_{10}) \arrow[d, phantom, "\cap" rotate=-90] \\
& \mathbb{R}^n = U_0 & U_1 = \mathbb{R}^n
\end{tikzcd}$$

$$\varphi_0^{-1}(W_{01}) = \{(a_1, \dots, a_n) : a_1 \neq 0\}$$

$$\varphi_1^{-1}(W_{10}) = \{(b_1, \dots, b_n) : b_1 \neq 0\}$$

$$\varphi_0(a_1, \dots, a_n) = (1, a_1, \dots, a_n) = \left(\frac{1}{a_1}, 1, \frac{a_2}{a_1}, \dots, \frac{a_n}{a_1}\right)$$

$$\varphi_1^{-1} \circ \varphi_0 (a_1, \dots, a_n) = \left(\frac{1}{a_1}, \frac{a_2}{a_1}, \dots, \frac{a_n}{a_1}\right)$$

differentiable where $a_1 \neq 0$. $\square$

Differentiable maps:

Definition: Given two differentiable manifolds M_1, M_2. A map $\varphi : M_1 \to M_2$ is differentiable if $\underline{\forall\, p \in M_1}$, $\exists\ U \subset \mathbb{R}^m$ $(m = \dim M_1)$ and $f : U \to M_1$ coordinate chart of M_1, $\exists\ V \subset \mathbb{R}^n$ $(n = \dim M_2)$ and $g : V \to M_2$ coordinate chart of M_2 s.t.

$$\left[\begin{array}{l} p \subset f(U), \quad \varphi(f(U)) \subset g(V) \\ \varphi(p) \subset g(V) \end{array}\right.$$

and $g^{-1}\varphi f$ is differentiable.

$$\begin{tikzcd}
\mathbb{R}^m \supset U \arrow[r, dashed, "g^{-1}\varphi f"] \arrow[d, "f"'] & V \subset \mathbb{R}^n \arrow[d, shift left, "g"] \\
f(U) \arrow[r, "\varphi"] \arrow[d, phantom, "\cap" rotate=90] & g(V) \arrow[u, shift left, "g^{-1}"] \arrow[d, phantom, "\cap" rotate=90] \\
M_1 \arrow[r, "\varphi"] & M_2
\end{tikzcd}$$

Note: With the above definition all coordinate charts $\varphi_\alpha : V_\alpha \to M$ are differentiable.

Definition: A differentiable curve in a differentiable manifold M is a differentiable map from an open interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ into M.

Definition: (tangent vectors) Suppose given a manifold M and a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M$. Let $\mathcal{D}$ be the set of differentiable functions on neighborhoods of $p := \alpha(0)$ in M to $\mathbb{R}$.

i.e., $\mathcal{D} := \left\{ f : \exists V \underset{\text{open}}{\subset} M, \text{ s.t. } p \in V \text{ and } f : V \to \mathbb{R} \text{ differentiable} \right\}$

The tangent vector to α at $p = \alpha(0)$ is the operator on $\mathcal{D}$ which sends $f \in \mathcal{D}$ to

$$\frac{d}{dt}(f\circ\alpha)\Big|_{t=0} \qquad t\in(-\varepsilon,\varepsilon)\xrightarrow{\alpha} M$$

$$\begin{tikzcd} M \arrow[d, phantom, "\cup"] & \\ V \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

if necessary, replace ε with a smaller positive real number so that $f\circ\alpha : (-\varepsilon,\varepsilon)\longrightarrow\mathbb{R}$ makes sense.

$$(-\varepsilon,\varepsilon) \subset \mathbb{R}$$

<u>notation</u>: $\alpha'(0) : \mathcal{D}\longrightarrow\mathbb{R}$ linear

$$\alpha'(0)(f) := \frac{d}{dt}(f\circ\alpha)\Big|_{t=0}$$

<u>Note</u>: A composition of differentiable maps is differentiable:

$$\begin{tikzcd} M_1 \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\varphi"'] & M_2 \arrow[r, "\psi"] & M_3 \end{tikzcd}$$

$$p \in M_1 \qquad \varphi(p) \qquad \psi(\varphi(p))$$

φ is differentiable: $\exists\ U \subset \mathbb{R}^l$, $f : U \longrightarrow M_1$
s.t. $p \in f(U)$

$\exists\ V \subset \mathbb{R}^m$ $g : V \longrightarrow M_2$ s.t. $\varphi(p) \in g(V)$

and $\varphi(f(U)) \subset g(V)$

$g^{-1}\varphi f$ is differentiable.

$$\begin{tikzcd}
\mathbb{R}^l \supset U \arrow[r, dotted, "g^{-1}\varphi f"] \arrow[d, "f"'] & V \subset \mathbb{R}^n \arrow[d, "g"] \\
f(U) \arrow[r, "\varphi"] & g(V) \\
\cap \arrow[r, "\varphi"] & \cap \\
M_1 & M_2
\end{tikzcd}$$

ψ is differentiable: $\exists\ V_1 \subset \mathbb{R}^m$, $g_1 : V_1 \longrightarrow M_2$ s.t.
$\varphi(p) \in g_1(V_1)$

$\exists\ W \subset \mathbb{R}^n$, $h : W \longrightarrow M_3$ s.t.
$\psi(g_1(V_1)) \subset h(W)$, $\psi(\varphi(p)) \in h(W)$

$h^{-1}\psi g$ is differentiable.

$$\begin{tikzcd}
\mathbb{R}^m \supset V_1 \arrow[r, dotted, "h^{-1}\psi g_1"] \arrow[d, "g_1"'] & W \subset \mathbb{R}^n \arrow[d, "h"] \\
g_1(V_1) \arrow[r, "\psi"] & h(W)
\end{tikzcd}$$

Shrink U so that $\psi \circ \varphi(f(U)) \subset W$

$\varphi(f(U)) \subset V$ shrink U so that $\varphi(f(U)) \subset V \cap V_1$

(replace U with the inverse image of $V \cap V_1$)

$$\begin{tikzcd} U \arrow[r, "g^{-1}\varphi f"] \arrow[rr, bend right] & V \cap V_1 \arrow[r, "h^{-1}\psi g_1"] & W \end{tikzcd}$$

after shrinking V and V_1 we can assume $g = g_1$

$$h^{-1}\psi g_1 g^{-1}\varphi f = h^{-1}\psi\varphi f$$

differentiable.