Shrink U so that $\psi \circ \varphi(f(U)) \subset W$

$\varphi(f(U)) \subset V$ shrink U so that $\varphi(f(U)) \subset V \cap V_1$

(replace U with the inverse image of $V \cap V_1$)

$$\begin{tikzcd} U \arrow[r, "g^{-1}\varphi f"] \arrow[rr, bend right] & V \cap V_1 \arrow[r, "h^{-1}\psi g_1"] & W \end{tikzcd}$$

$$h^{-1}\psi g_1 g^{-1} \varphi f = h^{-1}\psi\varphi f$$

differentiable.

After shrinking U, we can assume $g = g_1$ because

$h^{-1}\psi g = h^{-1}\psi g_1 g_1^{-1} g$ and $g_1^{-1} g$ is differentiable

because $V \longrightarrow M_2$ and $V_1 \longrightarrow M_2$ belong to

the same differentiable structure on M_2.

Definition: The tangent space to a differentiable manifold M at a point p is the set T_pM of all tangent vectors at p to differentiable curves $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ s.t. $\alpha(0) = p$.

Example: $M = \mathbb{R}^n$ $\quad p = (0, \dots, 0)$

$$\alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n \qquad \alpha(0) = p$$

Suppose given $U \subset \mathbb{R}^n$ open s.t. $U \ni p$ and

$$f : U \longrightarrow \mathbb{R} \quad \text{differentiable}$$

Denote $(x_1, \dots, x_n)$ coordinates on $\mathbb{R}^n$

write $\alpha(t) = (x_1(t), \dots, x_n(t))$, $t \in (-\varepsilon, \varepsilon)$

$$f(\alpha(t)) = f(x_1(t), \dots, x_n(t))$$

$$\frac{d}{dt}(f\circ\alpha)\Big|_{t=0} = \frac{d}{dt} f(x_1(t), \dots, x_n(t))$$

$$= \frac{\partial f}{\partial x_1}\Big|_p \cdot x_1'(0) + \frac{\partial f}{\partial x_2}\Big|_p x_2'(0) + \cdots + \frac{\partial f}{\partial x_n}\Big|_p x_n'(0)$$

This is the directional derivative of f in the direction $(x_1'(0), \dots, x_n'(0)) \in \mathbb{R}^n$ (vector)

= traditional velocity vector of α at 0.

More generally: Given M, $p \in M$, choose $V \subset \mathbb{R}^n$

$\varphi : V \longrightarrow M$ coordinate chart s.t. $p \in \varphi(V)$.

$\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ s.t. $\alpha(0) = p$.

take ε small enough so that the image of α is contained in $\varphi(V)$. $\forall$ $q \in \varphi(V)$

write $\varphi^{-1}(q) = (x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are coordinates on $\mathbb{R}^n \supset V$.

given f from a neighborhood of p to $\mathbb{R}$,

shrink U so that f is well-defined on $\varphi(U)$.

$$f : \varphi(U) \longrightarrow \mathbb{R}$$

$$\alpha'(0)(f) = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt}\big(f(\alpha(t))\big)\Big|_{t=0}$$

$$\alpha(t) \in \varphi(U) \qquad \varphi^{-1}(\alpha(t)) \in U$$

$$\varphi^{-1}(\alpha(t)) = (x_1(t), \dots, x_n(t))$$

$$(f \circ \alpha)(t) = f \circ \varphi \circ \varphi^{-1} \circ \alpha\,(t) = (f \circ \varphi)(x_1(t), \dots, x_n(t))$$

$$\begin{tikzcd} \mathbb{R}^n \supset U \arrow[r, "\varphi"] \arrow[rr, bend right] & \varphi(U) \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$\frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt}(f \circ \varphi)(x_1(t), \dots, x_n(t))\Big|_{t=0}$$

$$= \sum_{i=1}^{n} \frac{\partial (f \circ \varphi)}{\partial x_i}(0)\, x_i'(0)$$

So $\alpha'(0)$ has coordinates $(x_1'(0), \dots, x_n'(0))$
in the coordinate chart $\varphi : V \longrightarrow M$.

Definition: $\dfrac{\partial}{\partial x_i}$:= velocity vector at 0 to the

curve $\alpha(t) = \varphi(0, \dots, t, 0, \dots)$
$\uparrow$ i-th position.

$\dfrac{\partial}{\partial x_i} = (0, \dots, 1, 0, \dots)$ in the given coordinate system.

So $\alpha'(0) = \displaystyle\sum_{i=1}^{n} x_i'(0) \frac{\partial}{\partial x_i}$.

This shows that $T_p M \cong \mathbb{R}^n$ with basis $\left\{ \dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n} \right\}$

Differentials of maps: Given two manifolds M_1, M_2 and a differentiable map $a : M_1 \longrightarrow M_2$, the differential of a at a point $p \in M_1$ is the

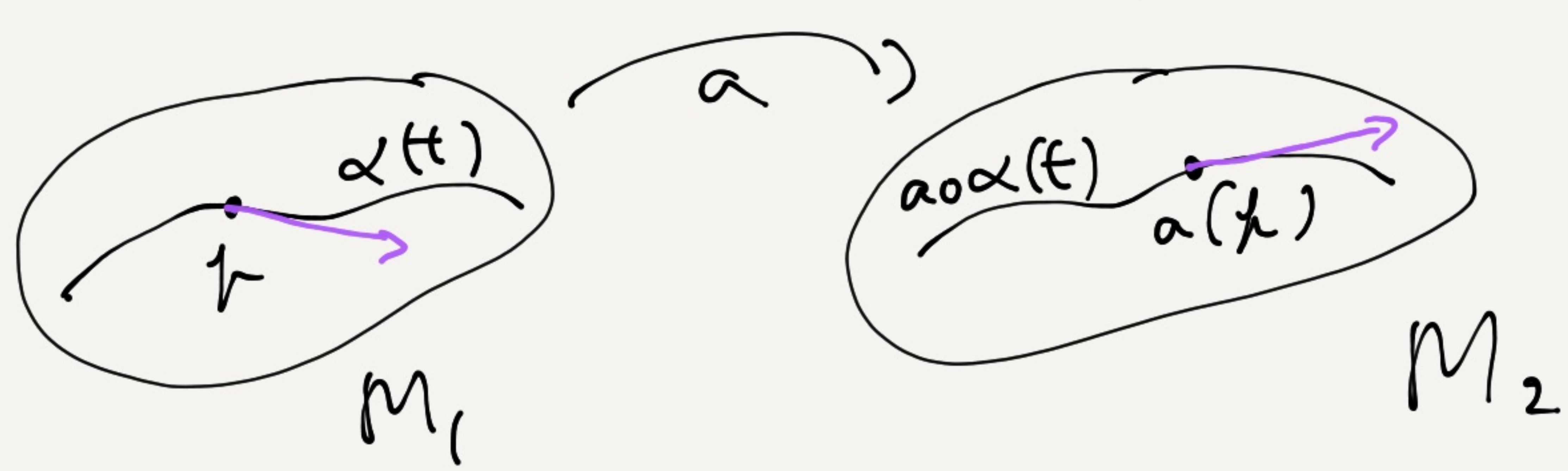
linear map $(da)_p : T_p M_1 \longrightarrow T_{a(p)} M_2$
defined as follows:

given $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M_1$ we define
$(da)_p(\alpha'(0)) \in T_{a(p)} M_2$ as the operator
which sends $g : V \longrightarrow \mathbb{R}$ $\quad V \subset M_2$ open neighborhood of $a(p)$.

to $$(da)_p(\alpha'(0))(g) := \frac{d}{dt}(g \circ a \circ \alpha)\Big|_{t=0}$$

or $$(da)_p(\alpha'(0))(g) = \alpha'(0)(g \circ a)$$

This means $(da)_p(\alpha'(0)) =$ tangent vector to $a \circ \alpha : (-\varepsilon, \varepsilon) \longrightarrow M_2$



Computation in coordinate charts:

$$\begin{array}{ccc} a : M_1 & \longrightarrow & M_2 \\ \varphi_1 \uparrow & & \varphi_2 \uparrow \\ \varphi_1^{-1}(\alpha(t)) \in U_1 \subset \mathbb{R}^m & & U_2 \subset \mathbb{R}^n \end{array}$$

$p \in M_1$ $\quad \alpha : (-\varepsilon, \varepsilon) \longrightarrow M_1$

$\alpha(0) = p$

$\alpha(-\varepsilon, \varepsilon) \subset \varphi_1(U_1)$

$$\varphi_1^{-1}\alpha(t) = (x_1(t), \cdots, x_m(t))$$

$$T_p M_1 = \left\langle \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m} \right\rangle = \mathbb{R}\frac{\partial}{\partial x_1} \oplus \cdots \oplus \mathbb{R}\frac{\partial}{\partial x_m}$$

$$T_{a(p)} M_2 = \left\langle \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n} \right\rangle$$ $(y_1, \cdots, y_n)$ coord. on U_2

$$g : U_2 \longrightarrow \mathbb{R}$$

$$(da)_p(\alpha'(0))(g) = \frac{d}{dt}(g \circ a \circ \alpha)\Big|_{t=0}$$

$$= \frac{d}{dt}(g \circ a \circ \varphi_1 \varphi_1^{-1} \circ \alpha)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(g\varphi_2\varphi_2^{-1}a\varphi_1\varphi_1^{-1}\alpha\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(g\varphi_2\varphi_2^{-1}a\varphi_1\right)\left(x_1(t),\dots,x_m(t)\right)\Big|_{t=0}$$

$$U_1 \xrightarrow{\varphi_1} \varphi_1(U_1) \xrightarrow{a} \varphi_2(U_2) \xrightarrow{\varphi_2^{-1}} U_2$$

write $\quad \varphi_2^{-1}a\varphi_1(x_1,\dots,x_m) = (y_1,\dots,y_n)$

$$y_i(x_1,\dots,x_m)$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial y_j}(g\varphi_2)\cdot\sum_{i=1}^{m}\frac{\partial y_j}{\partial x_i}\cdot x_i'(0)$$

$$= (da)_p(\alpha'(0))(g)$$

$$\Rightarrow \quad (da)_p(\alpha'(0)) = \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\partial y_j}{\partial x_i}(0)\,x_i'(0)\frac{\partial}{\partial y_j}$$

<u>as a matrix:</u> m vectors span $T_p M_1$

n " " $T_{a(p)} M_2$

$\Rightarrow$ the images of $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ give m column vectors, each of which has n entries.

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1}(0) & \frac{\partial y_1}{\partial x_2}(0) & \cdots \\ \frac{\partial y_2}{\partial x_1}(0) & \vdots & \\ \vdots & & \end{pmatrix} \quad \begin{array}{l} n \times m \\ \text{matrix} \end{array}$$

$$(d a)_p(\alpha'(0)) = \left(\frac{\partial y_j}{\partial x_i}(0)\right)_{\substack{1 \le j \le n \\ 1 \le i \le n}} \begin{pmatrix} x_1'(0) \\ \vdots \\ x_m'(0) \end{pmatrix}$$