Our goal: If a curve is length minimizing, then it is a geodesic, and, locally, geodesics are length minimizing (to be explained).

One of the important tools:

Gauss' lemma (Lemma 3.5):

Let $p \in M$, $v \in T_pM$ be such that $\exp_p(v)$ is well-defined.

Note: $\exp_p$: small ball in $T_pM \longrightarrow M$

$$d(\exp_p)_v : \underbrace{T_v(\text{ball})}_{\cong\, T_pM} \longrightarrow T_{\exp_p(v)}M$$

Then, $\forall\, w \in T_pM$, we have

$$\langle d(\exp_p)_v(v), d(\exp_p)_v(w) \rangle_{\exp_p(v)} = \langle v, w \rangle_p$$

Remark:

Recall $\exp_p(tv) = \gamma(1, p, tv) = \gamma(t, p, v)$
by the homogeneity lemma.

So the image of the line segment $\{tv \mid t \in \mathbb{R} \; -\varepsilon < t < \varepsilon\}$ is a geodesic in M through $p = \gamma(0, p, 0)$,

furthermore $t \longmapsto \gamma\left(t, p, \frac{v}{|v|}\right)$ is a geodesic with unit speed. So $\gamma\left(|v|, p, \frac{v}{|v|}\right) = \gamma(1, p, v) = \exp_p(v)$ is the point obtained by traveling a distance of $|v|$ along the geodesic $t \longmapsto \gamma\left(t, p, \frac{v}{|v|}\right)$.

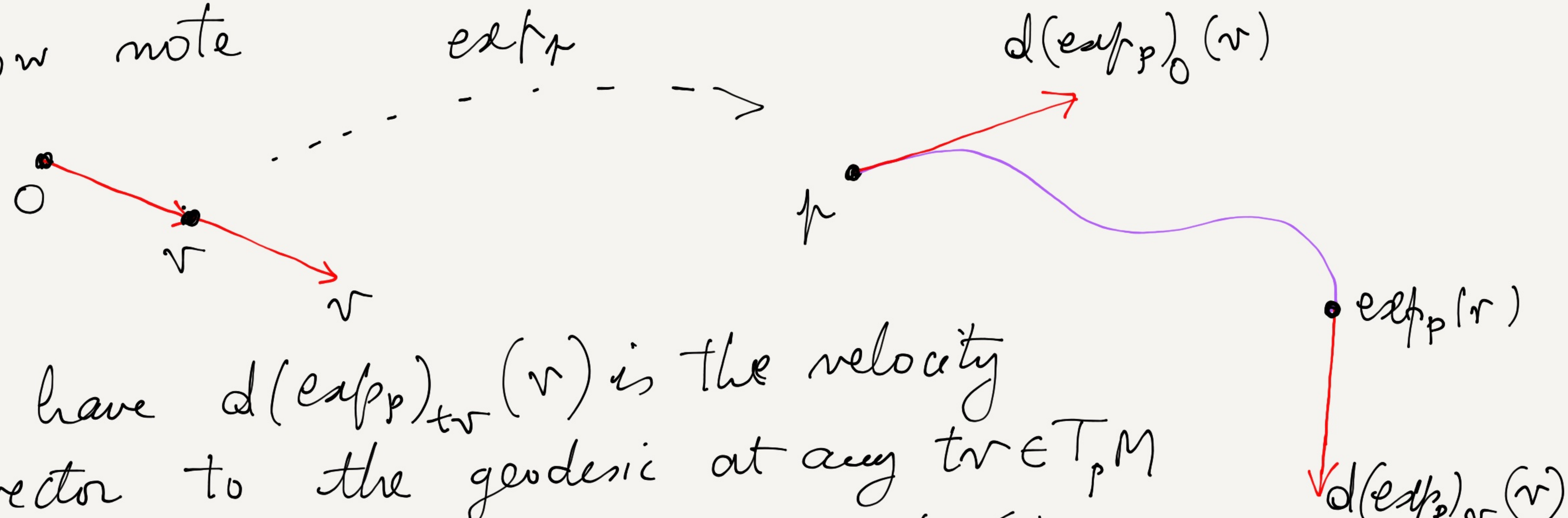
So length is preserved from lines through 0 in T_pM to radial geodesics in M.

Proof: Write $w = w_T + w_N$. We first prove

$$\langle d(\exp_p)_v(v), d(\exp_p)_v(w_T)\rangle_{\exp_p(v)} = \langle v, w_T\rangle_p$$

By linearity, it is sufficient to prove it for $w_T = v$

Now note



we have $d(\exp_p)_{tv}(v)$ is the velocity vector to the geodesic at any $tv \in T_pM$ $0 \le t \le 1$

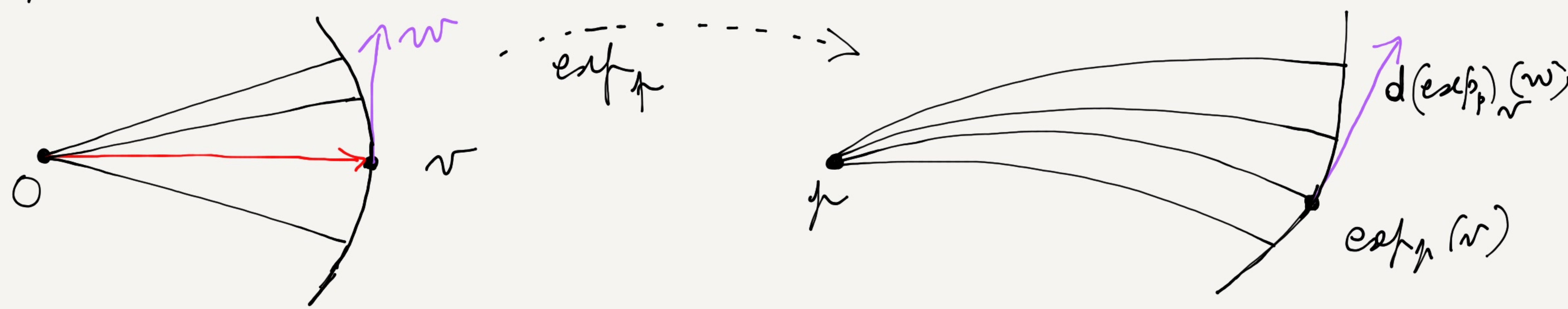
So $d(\exp_p)_{tv}(v)$ is the parallel transport of $d(\exp_p)_0(v)$ along the geodesic.

$$\Rightarrow \langle d(\exp_p)_v(v), d(\exp_p)_v(v)\rangle_{\exp_p(v)} = \langle v, v\rangle_p .$$

Now assume $w \perp v$, $w \neq 0$:

Choose a curve $s \mapsto v(s)$ in T_pM with $v(0)=v$, $v'(0)=w$, and $|v(s)| = $ constant, i.e., a circle in T_pM.

Since $\exp_p(v)$ is well-defined, $\exists\ \varepsilon>0$ s.t. $\exp_p(tv(s))$ is defined for $-\varepsilon < s < \varepsilon$, $0 \le t \le 1$.



Consider the parametrized surface:

$$f : \,]-\varepsilon, \varepsilon[\, \times [0,1] \longrightarrow M$$
$$(s,t) \longmapsto \exp_p(tv(s))$$

We have $\dfrac{\partial f}{\partial s} = d(\exp_p)_{tv(s)}(tv'(s))$

$$\frac{\partial f}{\partial t} = d(\exp_p)_{tv(s)}(v(s))$$

$$\Rightarrow \quad \frac{\partial f}{\partial s}(0,1) = d(\exp_p)_v(w) \quad , \quad \frac{\partial f}{\partial t}(0,1) = d(\exp_p)_v(v)$$

$$\Rightarrow \quad \langle d(\exp_p)_v(v), d(\exp_p)_v(w) \rangle_{\exp_p(v)} = \left\langle \frac{\partial f}{\partial s}(0,1), \frac{\partial f}{\partial t}(0,1) \right\rangle_{\exp_p(v)}$$

$$\frac{\partial f}{\partial s}(s,0) = 0 \qquad \frac{\partial f}{\partial t}(s,0) = d(\exp_p)_0(v(s))$$

We want to show $\langle d(\exp_p)_v(v), d(\exp_p)_v(w) \rangle = \langle v, w \rangle_p = 0$

Show $\left\langle \frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t) \right\rangle$ is independent of t

This will prove $\left\langle \frac{\partial f}{\partial s}(0,1), \frac{\partial f}{\partial t}(0,1) \right\rangle = \underbrace{\left\langle \frac{\partial f}{\partial s}(0,0), \frac{\partial f}{\partial t}(0,0) \right\rangle}_{=0}$

<u>Compute</u>: $\frac{\partial}{\partial t} \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle =$

$$= \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle$$

Now $t \longmapsto f(s,t) = \exp_p(t\,v(s))$ s fixed

is a radial geodesic at p and $\frac{\partial f}{\partial t}$ is its velocity

vector field $\Rightarrow$ $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$

Next, by the symmetry lemma: $\frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{ds}\frac{\partial f}{\partial t}$

and $\left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle = \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle = \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle$

$= 0$ because $\left|\frac{\partial f}{\partial t}\right| = \text{constant}$.

So $\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle$ is independent of t

and $0 = \left\langle \frac{\partial f}{\partial s}(0,0), \frac{\partial f}{\partial t}(0,0) \right\rangle = \left\langle \frac{\partial f}{\partial s}(0,1), \frac{\partial f}{\partial t}(0,1) \right\rangle$

$\quad\; \shortparallel$

$\langle v, w \rangle_p$

$= \langle d(\exp_p)_v(v), d(\exp_p)_v(w) \rangle$

at $\exp_p(v)$

$\square$

<u>Definition</u>: ① Radial geodesics at p are the images of lines through the origin in T_pM via $\exp_p$.

② A normal neighborhood of $p \in M$ is the image of a neighborhood V of $0 \in T_pM$ by $\exp_p$ s.t.

$$\exp_p\big|_V : V \longrightarrow \exp_p(V) \text{ is a diffeomorphism.}$$

③ If the open ball $B_\varepsilon(0) \subset T_pM$ satisfies $\overline{B_\varepsilon(0)} \subset V$ with V as in ②, then $\exp_p(B_\varepsilon(0))$ is the geodesic (or normal) open ball of radius ε at p, denoted $B_\varepsilon(p)$.

④ The geodesic sphere $S_\varepsilon(p)$ is the image of the sphere $S_\varepsilon(0) = \partial B_\varepsilon(0)$ by $\exp_p$.

Since $\overline{B_\varepsilon(0)} \subset V$ where $\exp_p$ is a diffeomorphism, $S_\varepsilon(p)$ is a submanifold of M of codimension 1, it is orthogonal to radial geodesics centered at p.