Our goal: If a curve is length minimizing, then it is a geodesic, and, locally, geodesics are length minimizing (to be explained).

One of the important tools:

**Gauss' Lemma (Lemma 3.5):**

Let \( p \in M \), \( v \in T_p M \) be such that \( \exp_p(v) \) is well-defined.

Note: \( \exp_p : \text{small ball in } T_p M \to M \)

\[ d(\exp_p)_v : T_v \text{(small)} \to T_{\exp_p(v)} M \]

Then, \( \forall w \in T_p M \), we have

\[ \langle d(\exp_p)_v(v), d(\exp_p)_v(w) \rangle_{\exp_p(v)} = \langle v, w \rangle_p \]
Remark:
Recall $\exp_p (-tv) = \gamma (1, p, tv) = \gamma (t, p, v)$ by the homogeneity lemma.

So the image of the line segment $\{ tv \mid t \in \mathbb{R}, -\varepsilon < t < \varepsilon \}$ is a geodesic in $M$ through $p = \gamma (0, p, 0)$.

Furthermore $t \mapsto \gamma (t, p, \frac{v}{|v|})$ is a geodesic with unit speed. So $\gamma (|v|, p, \frac{v}{|v|}) = \gamma (1, p, v) = \exp_p (v)$ is the point obtained by traveling a distance of $|v|$ along the geodesic $t \mapsto \gamma (t, p, \frac{v}{|v|})$.

So length is preserved from lines through $0$ in $T_p M$ to radial geodesics in $M$.

Proof: Write $w = w^T + w_N$. We first prove
\[
\langle d(\exp_p)_r (v), d(\exp_p)_r (w_T) \rangle = \langle v, w_T \rangle_{\exp_p (r)}
\]

By linearity, it is sufficient to prove it for \( w_T = r \).

\[\text{Now note } \exp_p \rightarrow d(\exp_p)_0 (v) \]

we have \( d(\exp_p)_r (v) \) is the velocity vector to the geodesic at any \( t \) in \( P \).

So \( d(\exp_p)_r (v) \) is the parallel transport of \( d(\exp_p)_0 (v) \) along the geodesic.

\[\Rightarrow \langle d(\exp_p)_r (v), d(\exp_p)_r (v) \rangle_{\exp_p (r)} = \langle v, v \rangle_{\exp_p (r)} .\]
Now assume \( w \perp v \), \( w \neq 0 \):

Choose a curve \( s \mapsto v(s) \) in \( T_pM \) with \( v(0) = v \), \( v'(0) = w \), and \( |v(s)| = \text{constant} \), i.e., a circle in \( T_pM \).

Since \( \exp_p(v) \) is well-defined, \( \exists \varepsilon > 0 \) s.t.
\[
\exp_p(tv(s)) \text{ is defined for } -\varepsilon < s < \varepsilon, \; 0 \leq t \leq 1.
\]

Consider the parametrized surface:
\[
f : \mathbb{R} - \varepsilon, \varepsilon \times \mathbb{R} \to M,
\]
\[
f(s, t) = \exp_p(tv(s))
\]

We have \( \frac{\partial f}{\partial s} = d(\exp_p)_{tv(s)}(tv'(s)) \)
\[ \frac{\partial}{\partial t} = \mathbf{d}(\exp^v)(x)(v) \]

\[ \Rightarrow \quad \frac{\partial}{\partial t}(0,1) = \mathbf{d}(\exp^v)(0)(v), \quad \frac{\partial}{\partial t}(0,1) = \mathbf{d}(\exp^v)(0)(v) \]

\[ \Rightarrow \quad \langle \mathbf{d}(\exp^v)(v), \mathbf{d}(\exp^v)(w) \rangle = \langle \frac{\partial}{\partial s}(0,1), \frac{\partial}{\partial t}(0,1) \rangle \]

\[ \frac{\partial}{\partial s}(0,0) = 0, \quad \frac{\partial}{\partial t}(0,0) = \mathbf{d}(\exp^v)(0)(v) \]

We want to show \( \langle \mathbf{d}(\exp^v)(v), \mathbf{d}(\exp^v)(w) \rangle = \langle v, w \rangle_p = 0 \)

Show \( \langle \frac{\partial}{\partial s}(0,t), \frac{\partial}{\partial t}(0,t) \rangle \) is independent of \( t \)

This will prove \( \langle \frac{\partial}{\partial s}(0,1), \frac{\partial}{\partial t}(0,1) \rangle = \langle \frac{\partial}{\partial s}(0,0), \frac{\partial}{\partial t}(0,0) \rangle = 0 \)

Compute: \[ \frac{\partial}{\partial t} \langle \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \rangle = \]
\[
= \left< \frac{D}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right> + \left< \frac{\partial f}{\partial s}, \frac{D}{dt} \frac{\partial f}{\partial t} \right>
\]

Now \( t \mapsto f(s,t) = \exp_p(tv(s)) \) is fixed if \( t \) is a radial geodesic at \( t \) and \( \frac{\partial f}{\partial t} \) is its velocity vector field \( \Rightarrow \frac{D}{dt} \frac{\partial f}{\partial t} = 0 \)

Next, by the symmetry lemma: \( \frac{D}{dt} \frac{\partial f}{\partial s} = \frac{D}{ds} \frac{\partial f}{\partial t} \)

and \( \left< \frac{D}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right> = \left< \frac{D}{ds} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right> = \frac{1}{2} \frac{\partial f}{\partial s} \left< \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right> = 0 \) because \( \left< \frac{\partial f}{\partial t} \right> = \text{constant} \).

\( \therefore \left< \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right> \) is independent of \( t \)

and \( 0 = \left< \frac{\partial f}{\partial s}(0,0), \frac{\partial f}{\partial t}(0,0) \right> = \left< \frac{\partial f}{\partial s}(0,1), \frac{\partial f}{\partial t}(0,1) \right> \)

\( \therefore \left< v, w \right> + \)

\( \left< d(\exp_p)_v(v), d(\exp_p)_v(w) \right> \) at \( \exp_p(v) \)
Definition: 1) Radial geodesics at \( p \) are the images of lines through the origin in \( T_p M \) via \( \exp_p \).

2) A normal neighborhood of \( p \in M \) is the image of a neighborhood \( V \) of \( 0 \in T_p M \) by \( \exp_p \), i.e., \( \exp_p |_V : V \rightarrow \exp_p(V) \) is a diffeomorphism.

3) If the open ball \( B_{e^p}(0) \subset T_p M \) satisfies \( B_{e^p}(0) \subset V \) with \( V \) as in 2), then \( \exp_p(B_{e^p}(0)) \) is the geodesic (or normal) open ball of radius \( e^p \) denoted \( B_e(\cdot) \).

4) The geodesic sphere \( S_e(\cdot) \) is the image of the sphere \( S_e(0) = \partial B_e(0) \) by \( \exp_p \).
Since $B_{\varepsilon}(0) \subset V$ where $\varepsilon > 0$ is a different planar $S_{\varepsilon}(x)$ is a sub-manifold of $M$ of codimension 1, it is orthogonal to radial geodesics centered at $x$. 