Since $B_\varepsilon(0) \subset V$ where $e_{xy}$ is a different frame, $S_\varepsilon(f)$ is a submanifold of $M$ of codimension 1, it is orthogonal to radial geodesics centered at $f$.

We now prove that geodesics are locally length-minimizing.

**Proposition:** \( \text{If } f \in M, U \text{ a normal neighborhood of } f, \\ B \subset U \text{ a geodesic ball centered at } f. \), \( \gamma : [0, 1] \rightarrow B \) a geodesic with \( \gamma(0) = f \) (i.e., \( \gamma \) is a radial geodesic).

If \( c : [0, 1] \rightarrow M \) is any piecewise differentiable curve with \( c(0) = \gamma(0) = f \), \( c(1) = \gamma(1) \), then \( l(c) \leq l(\gamma) \) \((l(c) = \text{length of } c)\), and equality holds iff \( c([0, 1]) = c([0, 1]) \).

**Proof:** First assume \( c([0, 1]) \subset B \).
$B \in \mathcal{C}$

Write $w(t) = \exp_r^{-1} \circ c(t) = n(t) \cdot v(t)$

where $n(t) = \frac{w(t)}{|w(t)|}$ and $|v(t)| = 1$

i.e., $n(t) = \frac{w(t)}{|w(t)|}$

$n : [0,1] \to \mathbb{R}$, $\phi : [0,1] \to T_p M$ are piecewise differentiable.

$\implies c(t) = \exp_r (n(t) \cdot v(t)) = \phi (n(t), t)$

where $\phi (n, t) := \exp_r (n \cdot v(t))$
\[ f : (0, 1) \times (0, 1) \rightarrow \mathbb{B} \\
(\xi, \tau) \quad \mapsto \exp(f_n(\tau)) \]

is a parametrized surface.

\[ l(c) = \int_0^1 |c'(t)| \, dt = \int_0^1 \left( \frac{dc}{dt} \right) \, dt \]

\[ \frac{dc}{dt} = \frac{d}{dt} f(n(\tau), \tau) = n(\tau) \frac{df}{dn} + \frac{df}{d\tau} \]

\[ = n(\tau) \frac{d}{d\tau} \exp(f_n(n(\tau))) (n(\tau)) + d(\exp(f_n))_{n(\tau)} (n(\tau)) \]

we know \[ |n(\tau)| = 1 \] so \[ \langle n(\tau), n'(\tau) \rangle = 0 \]

Yaus' Lemma \[ \Rightarrow \langle d(\exp(f_n))_{n(\tau)} (n(\tau)), d(\exp(f_n))_{n(\tau)} (n'(\tau)) \rangle = 0 \]

\[ \Rightarrow \langle \frac{df}{dn}, \frac{df}{d\tau} \rangle = 0 \]

\[ \Rightarrow \left\| \frac{dc}{dt} \right\|^2 = |n(\tau)|^2 \left| d(\exp(f_n))_{n(\tau)} (n(\tau)) \right|^2 + \left| d(\exp(f_n))_{n(\tau)} (n(\tau)) \right|^2 \]
If we have \( l(c) = l(\alpha) \), then:

\[
\frac{d}{dt} \left( \alpha(t) \right) = 0 \Rightarrow \alpha'(t) = 0 \ \forall \ t
\]

and

\[
|\alpha'(t)| = \alpha'(t) \Rightarrow \alpha'(t) \geq 0.
\]

So, \( \alpha(t) \) is constant \( \Rightarrow \ c(C_0, \pi) = \text{image of a line segment starting at } 0 \text{ in } T_M \) without doubling back.

\[
\Rightarrow \ \gamma(C_0, \pi) = c(C_0, \pi).
\]
If $c([0,1]) \subset B$, then let $t_1$ be the first value of $t \in (0,1)$ for which $c(t_1) \in \partial B = \partial S$, then

$$l(c) \geq l(c([0,t_1])) \geq l(\gamma_1 \text{ from } p \text{ to } c(t_1))$$

radius of $B \geq l(\gamma)$

\[ \text{Theorem and definition (Totally normal neighborhood).} \]

\[ \forall p \in M, \exists \delta > 0 \text{ and } W \text{ neighborhood of } p \]
\[ \forall q \in W \text{ early is a diffeomorphism on } B_{\delta}(0) \subset T_q M \]
and \( \exp_q(B_\epsilon(0)) \supseteq W \), i.e., \( W \) is a normal neighborhood of all of its points. We call \( W \) a totally normal neighborhood of \( p \) (or any of its points).

**Proof:** Recall Prop. 2.7: Given \( p \in M, \exists \nu \) neighborhood of \( p, \exists \epsilon > 0, \) s.t. if

\[
U := \{(q, w) \in TM : q \in \nu, w \in T_qM, |w| < \epsilon\}
\]

then \( \exists C \subseteq (-\epsilon, \epsilon) \times U \to M \) s.t.

\( \forall (q, w) \in U \) \( \gamma(t, q, w) \) is the unique geodesic with \( \gamma(0, q, w) = q \) and \( \gamma'(0, q, w) = w \).

**Define:** \( F : U \to M \times M \)

\[ (q, w) \mapsto (q, \exp_q(w)) \]

Show \( F \) is a local diffeomorphism near \((p, 0) \in W\).
$T (\omega) = T (\tau M) = T^\tau M \times T^\tau M$

$T_F (\sigma M) = T (\sigma M) = T^\tau M \times T^\tau M$

$(dF) (\eta, 0) = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ because $d (exp_\eta) = \text{Id}$.

$\Rightarrow \ F$ is a local diffeo near $(\eta, 0)$.

$\Rightarrow \ \exists \ \{ U' \subset U \ \text{s.t.} \ \ F \big|_{U'} : U' \rightarrow W'$

$W' \subset M \times M$

I shrink U' if necessary so that $\exists \ V' \subset M, \ \delta > 0 \ \text{s.t.}$

$U' = \{ (q, w) \in \tau M \mid q \in V', \ |w| < \delta \}$.

Choose $W \subset M$ s.t. $p \in W, \ n \times w \subset W'$. 
Claim: \( \varepsilon > 0 \) and \( W \) are what we are looking for:

If \( q \in W \), then \( B_{\varepsilon}(0) \subset T_q M \) and \( F(\{q\} \times B_{\varepsilon}(0)) = \{q\} \times W \)

\[ \exp_q(B_{\varepsilon}(0)) \]

recall:

\[ F: W' \cong W' \cup W \times W \]

\[ \{q\} \times B_{\varepsilon}(0) \]

\[ F(q,w) = (q, \exp_q(w)) \]

\[ \square \]