Since $\overline{B_\varepsilon(0)} \subset V$ where $\exp_p$ is a diffeomorphism, $S_\varepsilon(p)$ is a submanifold of M of codimension 1, it is orthogonal to radial geodesics centered at p.

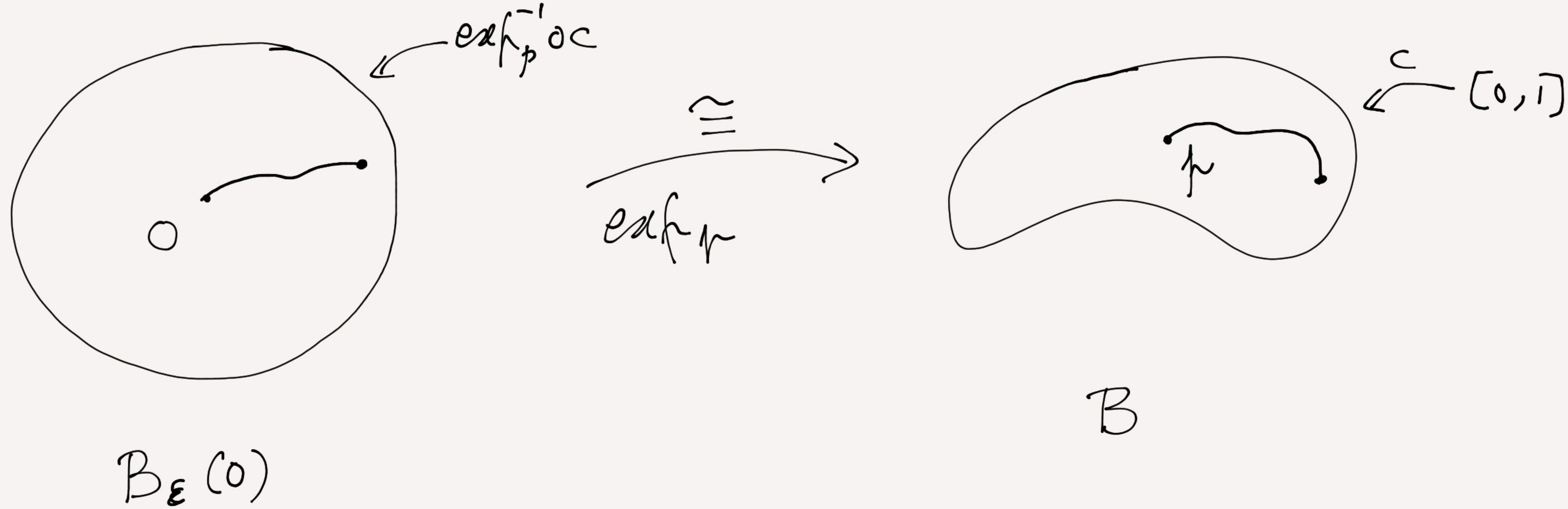
We now prove that geodesics are locally length-minimizing

<u>Proposition</u>: $p \in M$, U a normal neighborhood of p, $B \subset U$ a geodesic ball centered at p. $\gamma : [0,1] \to B$ a geodesic with $\gamma(0) = p$ (i.e., γ is a radial geodesic).

If $c : [0,1] \to M$ is any piecewise differentiable curve with $c(0) = \gamma(0) = p$, $c(1) = \gamma(1)$, then $l(\gamma) \leq l(c)$ ($l(c)$ = length of c), and equality holds iff

$$\gamma([0,1]) = c([0,1]).$$

<u>Proof</u>: First assume $c([0,1]) \subset B$.



Write $w(t) = \exp_p^{-1} \circ c(t) = r(t)\, v(t)$

where $r(t) = |w(t)|$ and $|v(t)| = 1$

i.e., $v(t) = \dfrac{w(t)}{|w(t)|}$

$r : [0,1] \longrightarrow \mathbb{R}$, $v : [0,1] \longrightarrow T_p M$

are piecewise differentiable.

$\Rightarrow c(t) = \exp_p (r(t)\, v(t)) = f(r(t), t)$

where $f(r,t) := \exp_p (r\, v(t))$

$$f : [0,1] \times [0,1] \longrightarrow B$$
$$(r,t) \longmapsto \exp_p(r v(t))$$

is a parametized surface.

$$l(c) = \int_0^1 |c'(t)|\, dt = \int_0^1 \left|\frac{dc}{dt}\right| dt$$

$$\frac{dc}{dt} = \frac{d}{dt} f(r(t),t) = r'(t)\frac{\partial f}{\partial r} + \frac{\partial f}{\partial t}$$
$$= r'(t)\, d(\exp_p)_{rv(t)}(v(t)) + d(\exp_p)_{rv(t)}(r v'(t))$$

we know $|v(t)|=1$, so $\langle v(t), v'(t)\rangle = 0$

Gauss' lemma $\Rightarrow \langle d(\exp_p)_{rv(t)}(v(t)), d(\exp_p)_{rv(t)}(v'(t))\rangle = 0$

$$\Rightarrow \quad \left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = 0$$

$$\Rightarrow \quad \left|\frac{dc}{dt}\right|^2 = |r'(t)|^2 \left| d(\exp_p)_{rv(t)}(v(t))\right|^2 + \left| d(\exp_p)_{rv(t)}(r v'(t))\right|^2$$

Gauss' lemma $\Rightarrow$ $\left| d(\exp_p)_{r v(t)} (v(t)) \right| = 1$

So $\left| \frac{dc}{dt} \right|^2 \geqslant |r'(t)|^2$

$$l(c) = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \left| \frac{dc}{dt} \right| dt \geqslant \lim_{\varepsilon \to 0} \int_\varepsilon^1 |r'(t)|\, dt \geqslant \lim_{\varepsilon \to 0} \int_\varepsilon^1 r'(t)\, dt$$

$$= \lim_{\varepsilon \to 0} (r(1) - r(\varepsilon)) = r(1) = l(\gamma)$$

If we have $l(c) = l(\gamma)$, then:

$d(\exp_p)_{r v(t)} (r v'(t)) = 0 \Rightarrow r v'(t) = 0 \quad \forall t$

and $|r'(t)| = r'(t) \Rightarrow r'(t) \geqslant 0$.

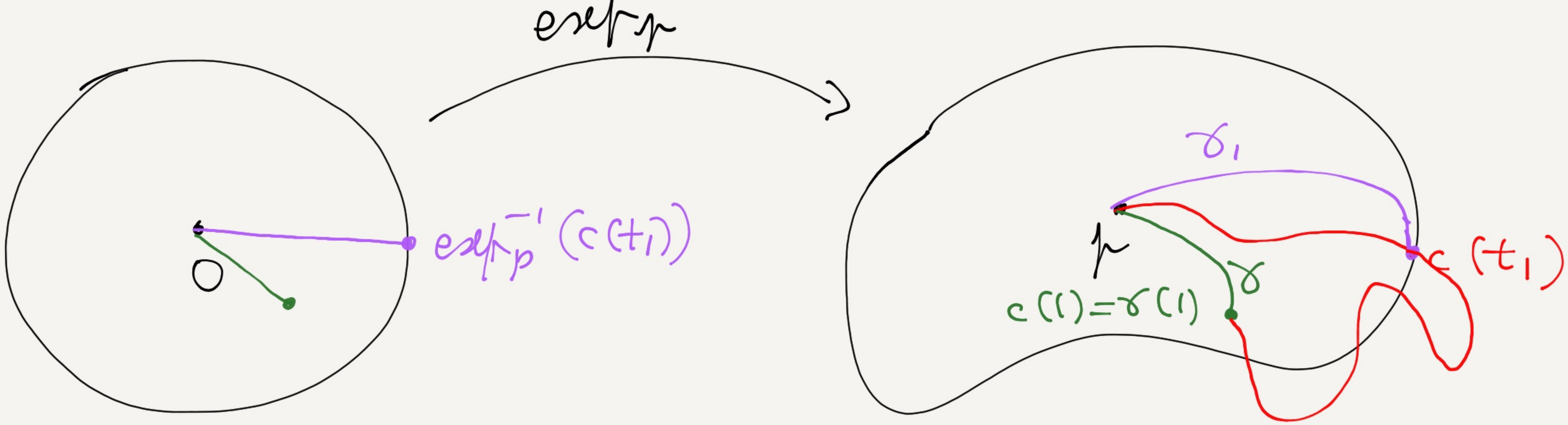
So $v(t)$ is constant $\Rightarrow$ $c([0,1])$ = image of a line segment starting at 0 in T_pM

$r' \geqslant 0 \Rightarrow c$ goes from $\gamma(0)$ to $\gamma(1)$ without doubling back.

$\Rightarrow \gamma([0,1]) = c([0,1])$.

If $c([0,1]) \not\subset B$, then let t_1 be the first value of $t \in (0,1)$ for which $c(t_1) \in \partial B = S$, then

$$l(c) \geq l\big(c([0,t_1])\big) \geq l(\gamma_1 \text{ from } p \text{ to } c(t_1))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad \text{radius of } B \geq l(\gamma)$$



$\square$

<u>Theorem and definition</u> (Totally normal neighborhood).

$\forall\, p \in M$, $\exists\ \delta > 0$ and W neighborhood of p s.t. $\forall\, q \in W$ $\exp_q$ is a diffeomorphism on $B_\delta(0) \subset T_qM$

and $\exp_q(B_\delta(0)) \supset W$, i.e., W is a normal neighborhood of all of its points. We call W a totally normal neighborhood of p (or any of its points).

Proof: Recall Prop. 2.7: Given $p \in M$, $\exists\ V$ neighborhood of p, $\exists\ \varepsilon > 0$, s.t. if

$$\mathcal{U} := \{ (q, w) \in TM : q \in V,\ w \in T_qM,\ |w| < \varepsilon \}$$

then $\exists$ C^∞ $\gamma : (-2, 2) \times \mathcal{U} \longrightarrow M$ s.t.

$\forall\ (q, w) \in \mathcal{U}$ $\gamma(t, q, w)$ is the unique geodesic with $\gamma(0, q, w) = q$ and $\gamma'(0, q, w) = w$.

Define:
$$F : \mathcal{U} \longrightarrow M \times M$$
$$(q, w) \longmapsto (q, \exp_q(w))$$

Show F is a locall diffeo. near $(p, 0) \in \mathcal{U}$.

$$T_{(p,0)}(\mathcal{U}) = T_{(p,0)}(TM) = T_pM \times T_pM$$

$$T_{F(p,0)}(M \times M) = T_{(p,p)}(M \times M) = T_pM \times T_pM$$

$$(dF)_{(p,0)} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$ because $d(\exp_p)_0 = Id.$

$\Rightarrow$ F is a local diffeo. near $(p,0)$.

$\Rightarrow \exists \begin{cases} \mathcal{U}' \subset \mathcal{U} \\ W' \subset M \times M \end{cases}$ s.t. $F\big|_{\mathcal{U}'} : \mathcal{U}' \longrightarrow W'$ is a diffeo.

Shrink $\mathcal{U}'$ if necessary so that $\exists V' \subset M$, $\delta > 0$ s.t.

$$\mathcal{U}' = \{(q,w) \in TM \mid q \in V', |w| < \delta\}.$$

Choose $W \subset M$ s.t. $p \in W$, $W \times W \subset W'$.

Claim: $\delta > 0$ and W are what we are looking for:

If $q \in W$, then $B_\delta(0) \subset T_qM$ and $F(\{q\} \times B_\delta(0)) \supset \{q\} \times W$

$$F(\{q\} \times B_\delta(0)) = \exp_q(B_\delta(0))$$

recall:

$$F : \mathcal{U}' \xrightarrow{\cong} W' \supset W \times W$$

$$\mathcal{U}' \supset \{q\} \times B_\delta(0) \qquad F(q,w) = (q, \exp_q(w)).$$

$\square$