Curvature: Suppose \( M \) is a manifold with an affine connection \( \nabla \).
Then the curvature \( R \) of \( \nabla \) is an operator (a tensor) which to a pair of vector fields \( X, Y \in \mathfrak{X}(M) \) associates a linear map \( R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M) \).

We define:
\[
R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z
\]
\[
\text{or } R(X,Y) := \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]}
\]
Linear in \( X, Y \) (with \( C^\infty \) coefficients).
Can check that it is also linear in \( Z \).

Introduce: \((X,Y,Z,T) := \langle R(X,Y)Z, T \rangle\)
when \( M \) is Riemannian and \( \nabla \) is the Levi-Civita connection.
\((X, Y, Z, T)\) is quadilinear.

It has symmetry properties such as:

\[
(X, Y, Z, T) = -(Y, X, Z, T)
\]

\[
= -(X, Y, T, Z)
\]

\[
= (Z, T, X, Y)
\]

\(R\) also satisfies the Bianchi identity:

\[
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0
\]

\underline{Curvature or sectional curvature:} \( \rho \in M \) Riemannian.

Given a two-dimensional subspace \( \sigma \subset T_x M \)

define the sectional curvature of \( \sigma \) as

\[
K(\sigma) := \frac{(X, Y, X, Y)}{|X \wedge Y|^2}
\]
Here $\{x, y\}$ is a basis of $\Sigma$

and $|x \langle y \rangle|^2 = |x|^2 |y|^2 - \langle x, y \rangle^2$

Note: If we choose an orthonormal basis of $\Sigma$, then

$\langle x, y \rangle = |x| |y| \cos \theta$

$\Rightarrow |x|^2 |y|^2 - \langle x, y \rangle^2 = |x|^2 |y|^2 - |x|^2 |y|^2 \cos^2 \theta$

$= |x|^2 |y|^2 \sin^2 \theta$

$= |x \times y|^2$ if we were in $\mathbb{R}^3$ and had a cross product.