Curvature: Suppose M is a manifold with an affine connection ∇.

Then the curvature R of ∇ is an operator (or a tensor) which to a pair of vector fields $X, Y \in \mathcal{X}(M)$ associates a linear map $R(X,Y): \mathcal{X}(M) \to \mathcal{X}(M)$

We define:

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

or $$R(X,Y) := \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]}$$

Linear in X, Y (with C^∞ coefficients)

Can check that it is also linear in Z.

Introduce: $(X,Y,Z,T) := \langle R(X,Y)Z, T \rangle$

when M is Riemannian and ∇ is the Levi-Civita connection.

(X, Y, Z, T) is quadrilinear.

It has symmetry properties such as:

$$
\begin{aligned}
(X, Y, Z, T) &= -(Y, X, Z, T) \\
&= -(X, Y, T, Z) \\
&= (Z, T, X, Y)
\end{aligned}
$$

R also satisfies the Bianchi identity:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

<u>Gaussian or sectional curvature</u>: $p \in M$ Riemannian.

Given a two-dimensional subspace $\sigma \subset T_pM$ define the sectional curvature of σ as

$$K(\sigma) := \frac{(x, y, x, y)}{|x \wedge y|^2}$$

here $\{x, y\}$ is a basis of σ

and $|x \wedge y|^2 := |x|^2|y|^2 - \langle x, y\rangle^2$

Note: If we choose an orthonormal basis of σ, then

$\langle x, y\rangle = |x||y| \cos\theta$ $\quad\quad$ $\theta =$ angle between x, y

so $\quad |x|^2|y|^2 - \langle x, y\rangle^2 = |x|^2|y|^2 - |x|^2|y|^2\cos^2\theta$

$= |x|^2|y|^2 \sin^2\theta$

$= |x \times y|^2$ if we were in $\mathbb{R}^3$ and had a cross product.