Math250A Riemannian Geometry

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1. Lie bracket

Lemma 5.2 *Let $X, Y \in \Gamma(M)$, then $[X, Y] \in \Gamma(M)$.*

Proof. Given $p \in M$, let $x : U \to \mathbb{R}^m$ be a coordinate chart. Express X, Y by the following:

$$X = a^i \frac{\partial}{\partial x^i}, \qquad Y = b^j \frac{\partial}{\partial x^j}$$

Then for any differentiable function f, we have

$$\begin{aligned} XYf &= X(b^j \frac{\partial f}{\partial x^j}) = a^i \frac{\partial}{\partial x^i}(b^j \frac{\partial f}{\partial x^j}) \\ &= a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} + a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j} \end{aligned}$$

And

$$\begin{aligned} YXf &= Y(a^i \frac{\partial f}{\partial x^i}) = b^j \frac{\partial}{\partial x^i}(a^i \frac{\partial f}{\partial x^i}) \\ &= b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i} + a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j} \end{aligned}$$

So

$$\begin{aligned} [X, Y]f &= (XY - YX)f \\ &= a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} + a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i} - a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j} \\ &= (a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}) \frac{\partial f}{\partial x^j} \end{aligned}$$

That is to say, $[X, Y] = (a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}) \frac{\partial}{\partial x^j} \in \Gamma(M)$. This holds for each coordinates of M, so it defines on M. Also note the definition is independent of charts, so it is unique. $\square$

5.3 Proposition. *If X, Y and Z are differentiable vector fields on M, a, b are real numbers, and f, g are differentiable functions, then:*

(a) $[X, Y] = -[Y, X]$ *(anticommutativity),*
(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *(linearity),*
(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacobi identity),*
(d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

Proof. (a) and (b) are immediate. In order to prove (c), it suffices to observe that, on the one hand,

$$[[X,Y],Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$$

while, on the other hand,

$$[X,[Y,Z]] + [Y,[Z,X]]$$

$$= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY.$$

Because the second members of the expressions above are equal, (c) follows using (a).

Finally, to prove (d), calculate

$$[fX, gY] = fX(gY) - gY(fX) = fgXY + fX(g)Y$$
$$- gfYX - gY(f)X = fg[X,Y] + fX(g)X - gY(f)X. \square$$

2. Another interpretation of Lie bracket

Next, we will study another way to think about the bracket by "trajectories".

Definition: *A curve* $\alpha : (-\delta, \delta) \to M$ *where* $\alpha'(t) := da(\frac{d}{dt}) = X_{\alpha(t)}$ *and* $\alpha(0) = q$ *is called a trajectory of the field* X *that passes through* q *for* $t = 0$.

Theorem: (Fact from ODEs) *Given* $X \in \Gamma(M)$ *and* $p \in M$. *Then there exist a nbh* $U \subset M$ *of* p, $\delta > 0$ *and a differentiable mapping* $\varphi : (-\delta, \delta) \times U \to M$ *s.t. the curve is the unique that satisfies* $\frac{\partial \varphi}{\partial t} = X_{\varphi(t,q)}$ *and* $\varphi(0,q) = q$.

Locally speaking, take coordinates $x = (x^1, \ldots, x^n)$. Write $X = a^i(x)\frac{\partial}{\partial x^i}$, and $x(t) := x(\varphi_t(q)) = x(\alpha_q(t))$. Then

$$\frac{dx}{dt}(t) = (a^1(x(t)), \ldots, a^n(x(t)))$$

With initial condition $x(0) = x(q)$. We can solve this by using the Picard's Theorem.

5.4 Proposition. *Let* X, Y *be differentiable vector fields* on a differentiable manifod M, let $p \in M$, and let φ_t be the local flow of X in a neighborhood U of p. Then

$$[X,Y]_p = \lim_{t \to 0} \frac{1}{t}[Y - d\varphi_t Y]_{\varphi_t(p)}$$

For the proof, we need the following lemma from calculus.

5.5 Lemma. *Let* $h: (-\delta, \delta) \times U \to \mathbf{R}$ *be a differentiable mapping with* $h(0,q) = 0$ *for all* $q \in U$. *Then there exists a differentiable mapping* $g: (-\delta, \delta) \times U \to \mathbf{R}$ *with* $h(t,q) = tg(t,q)$; *in particular,*

$$g(0,q) = \left.\frac{\partial h(t,q)}{\partial t}\right|_{t=0}.$$

Proof of lemma. It suffices to define, for fixed t,

$$g(t,q) = \int_0^1 \frac{\partial h(ts,q)}{\partial(ts)}\, ds$$

and, after changing variables, observe that

$$\boxed{tg(t,q) = \int_0^t \frac{\partial h(ts,q)}{\partial(ts)} d(ts) = h(t,q).}$$

Proof of the Proposition. Let f be a differentiable function in a neighborhood of p. Putting

$$\boxed{h(t,q) = f(\varphi_t(q)) - f(q),}$$

and applying the lemma we obtain a differentiable function $g(t,q)$ such that

$$f \circ \varphi_t(q) = f(q) + tg(t,q) \quad \text{and} \quad g(0,q) = Xf(q).$$

Accordingly

$$((d\varphi_t Y)f)(\varphi_t(p)) = (Y(f \circ \varphi_t))(p) = Yf(p) + t(Yg(t,p)).$$

Therefore

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}[Y - d\varphi_t Y]f(\varphi_t p) &= \lim_{t\to 0} \frac{(Yf)(\varphi_t p) - Yf(p)}{t} - (Yg(0,p)) \\ &= (X(Yf))(p) - (Y(Xf))(p) \\ &= ((XY - YX)f)(p) = ([X,Y]f)(p). \quad \square\end{aligned}$$

3. Partition of Unity

Over a point $p \in V$ where V is a coordinate neighborhood diffeomorphic to an open ball, We can construct a bump function, i.e., $0 \le f \le 1$ and

$$f(q) = \begin{cases} 1 & q \in \bar{U} \\ 0 & q \notin V \end{cases}$$

where $\bar{U} \subset V$.

Idea: Given $p \in \mathbb{R}^n$ and and ball $B_r(p)$, Let $U = B_s(p)$ for $s = r/2$, and define $f(q) = \beta(-2|p-q|/r)$ for $q \in \mathbb{R}^n$, where $\beta : \mathbb{R} \to \mathbb{R}$ is given by

$$\beta(t) = \frac{\int_{-\infty}^t \alpha(s)ds}{\int_{-2}^{-1} \alpha(s)ds}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is the smooth function

$$\alpha(t) = \begin{cases} \exp(-\frac{1}{(t+2)(-1-t)}) & t \in [-2,-1] \\ 0 & \text{Otherwise} \end{cases}$$

A simple computation shows that f defines above is a bump function.

Remark: This arises the idea of Partition of Unity.

Let M be a differentiable manifold. A family of open sets $V_\alpha \subset M$ with $\bigcup_\alpha V_\alpha = M$ is said to be *locally finite* if every point $p \in M$ has a neighborhood W such that $W \cap V_\alpha \neq \phi$ for only a finite number of indices. The *support* of a function $f: M \to \mathbf{R}$ is the closure of the set of points where f is different from zero.

We say that a family $\{f_\alpha\}$ of differentiable functions $f_\alpha: M \to \mathbf{R}$ is a *differentiable partition of unity* if:

(1) For all $\alpha, f_\alpha \geq 0$ and the support of f_α is contained in a coordinate neighborhood $V_\alpha = \mathbf{x}_\alpha(U_\alpha)$ of a differentiable structure $\{(U_\beta, \mathbf{x}_\beta)\}$ of M.

(2) The family $\{V_\alpha\}$ is locally finite.

(3) $\sum_\alpha f_\alpha(p) = 1$, for all $p \in M$ (this condition makes sense, because for each p, $f_\alpha(p) \neq 0$ only for a finite number of indices).

It is customary to say that the partition of unity $\{f_\alpha\}$ is *subordinate to the covering* $\{V_\alpha\}$.

5.6 Theorem. *A differentiable manifold M has a differentiable partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.*

Proof. **(sketched)** First to show it is Hausdorff. Given two points $m, m' \in M$. There exists ϕ_a s.t. $\phi_a(m) \neq 0$. So m in a coordinate chart U which contains the support of ϕ_a. If $m' \in U$, notice that U is diffeomorphic to an open subset of $\mathbb{R}^n$, which is Hausdorff, so admits disjoint open neighborhoods. If $m' \notin U$, $\phi_a(m') = 0$. Because ϕ_a is continuous, preimage of open sets are open.

Next show it is second countable.

Lemma: If $\{U_a\}$ is a chart covering of a connected manifold M. If for each a, there are only countable many b s.t. $U_a \cap U_b \neq \emptyset$, then M has a countable basis.

(Proof of lemma:) Note that if M has a countable atlas, then it is second countable. So for the atlas $\{U_a\}$, we suffice to show that it admits a countable subcovering. Now let B_1 is just one of $\{U_a\}$. And let B_2 as the union of the sets of $\{U_a\}$ with a non-empty intersection with B_1. Inductively, we have $B_1 \subset B_2 \ldots$. Denote $B = \cup B_n$. We claim that $B = M$. I.e., $\{B_n\}$ is a countable subcovering. So the lemma is proved.

Now let $\{\phi_a\}$ be a partition of unity on M and M' a component of M. Denote $C_a = \operatorname{supp}\phi_a$ and because it is locally finite, we have $C_a \cap C_b = \emptyset$ except for a finite number of supports C_b. Denote $W_a := \{\phi_a > 0\} \subset U_a$ (so it is also a coordinate domain), and $V_a = W_a \cap M'$. So V_a is either $\emptyset$ or a coordinate domain of M'. Note that $\{V_a\}$ cover M' and $V_a \subset W_a \subset C_a$, $V_a \cap V_b = \emptyset$ except for a finite number of sets V_b. So by the lemma it admits a countable basis.

Let's only prove the easier case that M is compact. For each $q \in M$, let $U_q \in \{U_a\}$ and ψ_q be a bump function s.t. $\operatorname{supp}\psi_q \subset U_q$. There is a neighborhood W_q of q s.t. $\psi_q(s) > 0$ for $s \in W_q$. So $\{W_q\}$ is a open cover of M. Find a finite subcover $\{W_{q_1}, \ldots, W_{q_n}\}$. Define $\varphi_j = \frac{\psi_{q_j}}{\sum \psi_{q_i}}$. We claim this $\{\varphi_j\}$ is a partition of unity.

Finally, choose index $\tau(j)$ s.t. $\operatorname{supp}(\varphi_j) \subset U_{\tau(j)}$ for $j = 1, \ldots, n$. Now we define $f_a = \sum\limits_{\tau(j)=a} \varphi_j$, and we directly have

$$\sum_{\tau(j)=a} f_a = \sum_a \sum_{\tau(j)=a} \varphi_j = \sum_n^{j=1} \varphi_j = 1$$

We claim that $\{f_a\}$ is subordinate to $\{U_a\}$. $\square$

Remark: If we require each f_a has compactly support, then the theorem becomes: For a open cover $\{U_a\}$, there exists $\{f_n\}$ smooth partition of unity s.t. each support is compact and there exists U_a s.t. $\operatorname{supp} f_n \subset U_a$. Why compactness matters? Consider $M = (0,1)$ with $U_1 = (0, 0.8)$ and $U_2 = (0.2, 1)$. Then $\operatorname{supp} f_1 + f_2$ is also compact, but $M = (0,1)$ is not compact, contradiction.

References

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