1. Lie bracket

Lemma 5.2 Let \( X, Y \in \Gamma(M) \), then \( [X, Y] \in \Gamma(M) \).

Proof. Given \( p \in M \), let \( x : U \to \mathbb{R}^m \) be a coordinate chart. Express \( X, Y \) by the following:

\[
X = a^i \frac{\partial}{\partial x^i}, \quad Y = b^j \frac{\partial}{\partial x^j}
\]

Then for any differentiable function \( f \), we have

\[
XYf = X(b^j \frac{\partial f}{\partial x^j}) = a^i \frac{\partial}{\partial x^i}(b^j \frac{\partial f}{\partial x^j})
\]

\[
= a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} + a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j}
\]

And

\[
YXf = Y(a^i \frac{\partial f}{\partial x^i}) = b^j \frac{\partial}{\partial x^j}(a^i \frac{\partial f}{\partial x^i})
\]

\[
= b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i} + a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j}
\]

So

\[
[X, Y]f = (XY - YX)f
\]

\[
= a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} + a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i} - a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j}
\]

\[
= (a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}) \frac{\partial f}{\partial x^j}
\]

That is to say, \( [X, Y] = (a^i \frac{\partial b^j}{\partial x^i} - b^j \frac{\partial a^i}{\partial x^i}) \frac{\partial}{\partial x^j} \in \Gamma(M) \). This holds for each coordinates of \( M \), so it defines on \( M \). Also note the definition is independent of charts, so it is unique. \( \square \)

5.3 Proposition. If \( X, Y \) and \( Z \) are differentiable vector fields on \( M \), \( a, b \) are real numbers, and \( f, g \) are differentiable functions, then:

(a) \( [X, Y] = -[Y, X] \) (anticommutativity),

(b) \( [aX + bY, Z] = a[X, Z] + b[Y, Z] \) (linearity),

(c) \( [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \) (Jacobi identity),

(d) \( [fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X \).
Another interpretation of Lie bracket

Next, we will study another way to think about the bracket by "trajectories".

**Definition:** A curve \( \alpha : (-\delta, \delta) \to M \) where \( \alpha'(t) := \frac{d\alpha(t)}{dt} = X_{\alpha(t)} \) and \( \alpha(0) = q \) is called a trajectory of the field \( X \) that passes through \( q \) for \( t = 0 \).

**Theorem:** (Fact from ODEs) Given \( X \in \Gamma(M) \) and \( p \in M \). Then there exist a nbh \( U \subset M \) of \( p \), \( \delta > 0 \) and a differentiable mapping \( \varphi : (-\delta, \delta) \times U \to M \) s.t. the curve is the unique that satisfies \( \frac{\partial \varphi}{\partial t} = X_{\varphi(t,q)} \) and \( \varphi(0,q) = q \).

Locally speaking, take coordinates \( x = (x^1, \ldots, x^n) \). Write \( X = a^i(x) \frac{\partial}{\partial x^i} \), and \( x(t) := x(\varphi_t(q)) = x(\alpha_q(t)) \). Then

\[
\frac{dx}{dt}(t) = (a^1(x(t)), \ldots, a^n(x(t)))
\]

With initial condition \( x(0) = x(q) \). We can solve this by using the Picard’s Theorem.

**5.4 Proposition.** Let \( X, Y \) be differentiable vector fields on a differentiable manifold \( M \), let \( p \in M \), and let \( \varphi_t \) be the local flow of \( X \) in a neighborhood \( U \) of \( p \). Then

\[
[X, Y]_p = \lim_{t \to 0} \frac{1}{t}[Y - d\varphi_t Y]_{\varphi_t(p)}
\]

For the proof, we need the following lemma from calculus.

**5.5 Lemma.** Let \( h: (-\delta, \delta) \times U \to \mathbb{R} \) be a differentiable mapping with \( h(0,q) = 0 \) for all \( q \in U \). Then there exists a differentiable mapping \( g: (-\delta, \delta) \times U \to \mathbb{R} \) with \( h(t,q) = tg(t,q) \); in particular,

\[
g(0,q) = \frac{\partial h(t,q)}{\partial t} \bigg|_{t=0}.
\]
Proof of lemma. It suffices to define, for fixed $t$,
\[ g(t, q) = \int_0^1 \frac{\partial h(ts, q)}{\partial (ts)} ds \]
and, after changing variables, observe that
\[ tg(t, q) = \int_0^t \frac{\partial h(ts, q)}{\partial (ts)} d(ts) = h(t, q). \]

Proof of the Proposition. Let $f$ be a differentiable function in a neighborhood of $p$. Putting
\[ h(t, q) = f(\varphi_t(q)) - f(q), \]
and applying the lemma we obtain a differentiable function $g(t, q)$ such that
\[ f \circ \varphi_t(q) = f(q) + tg(t, q) \quad \text{and} \quad g(0, q) = Xf(q). \]
Accordingly
\[ ((d\varphi_tY)f)(\varphi_t(p)) = (Y(f \circ \varphi_t))(p) = Yf(p) + t(Yg(t, p)). \]

Therefore
\[
\lim_{t \to 0} \frac{1}{t} [Y - d\varphi_tY]f(\varphi_t(p)) = \lim_{t \to 0} \frac{(Yf)(\varphi_t(p)) - Yf(p)}{t} - (Yg(0, p))
= (XYf)(p) - (Y(Xf))(p)
= ((XY - YX)f)(p) = (\{X, Y\}f)(p). \quad \square
\]

3. Partition of Unity

Over a point $p \in V$ where $V$ is a coordinate neighborhood diffeomorphic to an open ball, We can construct a bump function, i.e., $0 \leq f \leq 1$ and
\[ f(q) = \begin{cases} 
1 & q \in \bar{U} \\
0 & q \notin V 
\end{cases} \]
where $\bar{U} \subset V$.

Idea: Given $p \in \mathbb{R}^n$ and and ball $B_r(p)$, Let $U = B_s(p)$ for $s = r/2$, and define $f(q) = \beta(-2|p - q|/r)$ for $q \in \mathbb{R}^n$, where $\beta : \mathbb{R} \to \mathbb{R}$ is given by
\[ \beta(t) = \frac{\int_{-\infty}^{t} \alpha(s) ds}{\int_{-2}^{1} \alpha(s) ds} \]
where \( \alpha : \mathbb{R} \to \mathbb{R} \) is the smooth function

\[
\alpha(t) = \begin{cases} 
\exp\left(\frac{-1}{(t+2)(1-t)}\right) & t \in [-2, -1] \\
0 & \text{Otherwise}
\end{cases}
\]

A simple computation shows that \( f \) defines above is a bump function.

**Remark:** This arises the idea of Partition of Unity.

**5.6 Theorem.** A differentiable manifold \( M \) has a differentiable partition of unity if and only if every connected component of \( M \) is Hausdorff and has a countable basis.

**Proof.** (sketched) First to show it is Hausdorff. Given two points \( m, m' \in M \) in \( M \), there exists \( \phi_a \) s.t. \( \phi_a(m) \neq 0 \). So \( m \) in a coordinate chart \( U \) which contains the support of \( \phi_a \). If \( m' \in U \), notice that \( U \) is diffeomorphic to an open subset of \( \mathbb{R}^n \), which is Hausdorff, so admits disjoint open neighborhoods. If \( m' \notin U \), \( \phi_a(m') = 0 \). Because \( \phi_a \) is continuous, preimage of open sets are open.

Next show it is second countable.

**Lemma:** If \( \{U_a\} \) is a chart covering of a connected manifold \( M \). If for each \( a \), there are only countable many \( b \) s.t. \( U_a \cap U_b \neq \emptyset \), then \( M \) has a countable basis.

(Proof of lemma) Note that if \( M \) has a countable atlas, then it is second countable. So for the atlas \( \{U_a\} \), we suffice to show that it admits a countable subcovering. Now let \( B_1 \) is just one of \( \{U_a\} \). And let \( B_2 \) as the union of the sets of \( \{U_a\} \) with a non-empty intersection with \( B_1 \). Inductively, we have \( B_1 \subseteq B_2 \ldots \). Denote \( B = \bigcup B_a \). We claim that \( B = M \). I.e., \( \{B_a\} \) is a countable subcovering. So the lemma is proved.
Now let \( \{\phi_a\} \) be a partition of unity on \( M \) and \( M' \) a component of \( M \). Denote \( C_a = \text{supp} \phi_a \) and because it is locally finite, we have \( C_a \cap C_b = \emptyset \) except for a finite number of supports \( C_b \). Denote \( W_a := \{\phi_a > 0\} \subset U_a \) (so it is also a coordinate domain), and \( V_a = W_a \cap M' \). So \( V_a \) is either \( \emptyset \) or a coordinate domain of \( M' \). Note that \( \{V_a\} \) cover \( M' \) and \( V_a \subset W_a \subset C_a \), \( V_a \cap V_b = \emptyset \) except for a finite number of sets \( V_b \). So by the lemma it admits a countable basis.

Let’s only prove the easier case that \( M \) is compact. For each \( q \in M \), let \( U_q \in \{U_a\} \) and \( \psi_q \) be a bump function s.t. \( \text{supp} \psi_q \subset U_q \). There is a neighborhood \( W_q \) of \( q \) s.t. \( \psi_q(s) > 0 \) for \( s \in W_q \). So \( \{W_q\} \) is an open cover of \( M \). Find a finite subcover \( \{W_{q_1}, \ldots, W_{q_n}\} \). Define \( \varphi_j = \frac{\psi_{q_j}}{\sum \psi_{q_i}} \). We claim this \( \{\varphi_j\} \) is a partition of unity.

Finally, choose index \( \tau(j) \) s.t. \( \text{supp}(\varphi_j) \subset U_{\tau(j)} \) for \( j = 1, \ldots, n \). Now we define \( f_a = \sum_{\tau(j)=a} \varphi_j \), and we directly have

\[
\sum_{\tau(j)=a} f_a = \sum_a \sum_{\tau(j)=a} \varphi_j = \sum_{a} \varphi_j = 1
\]

We claim that \( \{f_a\} \) is subordinate to \( \{U_a\} \). \( \square \)

**Remark:** If we require each \( f_a \) has compactly support, then the theorem becomes: For a open cover \( \{U_a\} \), there exists \( \{f_a\} \) smooth partition of unity s.t. each support is compact and there exists \( U_a \) s.t. \( \text{supp} f_a \subset U_a \). Why compactness matters? Consider \( M = (0, 1) \) with \( U_1 = (0, 0.8) \) and \( U_2 = (0.2, 1) \). Then \( \text{supp} f_1 + f_2 \) is also compact, but \( M = (0, 1) \) is not compact, contradiction.

**References**


