Examples of differentiable manifold

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1 tangent bundle

Given $M$ a manifold of dimension $n$ with charts $\{(X_\alpha, U_\alpha)\}$, want to construct a differentiable manifold $TM$ for which $M$ is a submanifold. The idea of construction would be attaching every point with its tangent space.

Def 2.6 gives the idea of parameterization.

The coordinate charts is $\{(Y_\alpha, U_\alpha \times \mathbb{R}^n)\}$:

$$Y_\alpha(x_1, ..., x_n, u_1, ..., u_n) = (X_\alpha(x_1, ..., x_n), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}).$$

Proof 1.1 (TM is a differentiable manifold) $(p, v) \in TM$, $\exists U_\alpha, X_\alpha$, such that $p \in X_\alpha(U_\alpha)$. Then $(p, v) \in Y_\alpha(U_\alpha, \mathbb{R}^n)$, i.e. $\{(Y_\alpha, U_\alpha \times \mathbb{R}^n)\}$ covers $TM$.

Injectivity of $Y_\alpha$ follows from our construction. It remains to show that the charts are compatible. $Y_\alpha(x_1, ..., x_n, u_1, ..., u_n) = Y_\beta(x_1^*, ..., x_n^*, u_1^*, ..., u_n^*)$ would imply $(x_1, ..., x_n) = X_\alpha^{-1}X_\beta(x_1^*, ..., x_n^*)$, $(u_1, ..., u_n) = dX_\alpha^{-1}X_\beta(u_1^*, ..., u_n^*)$. Thus, $Y_\alpha^{-1}Y_\beta$ is differentiable from $U_\beta \times \mathbb{R}^n$ to $U_\alpha \times \mathbb{R}^n$.

2 Regular manifold in $\mathbb{R}^n$

Regular manifold $M$ in $\mathbb{R}^n$ is a generalization of regular surface in $\mathbb{R}^3$. Firstly, we are given differentiable maps $X_\alpha$ from $U_\alpha \subset \mathbb{R}^m$ to $M$. More we require $(dX_\alpha)_p$ to be injective from $\mathbb{R}^m$ to $T_pM$. This require that $X_\alpha$ is an immersion from $U_\alpha$ to $M$, and it require that $m \leq n$. The proof that $\{X_\alpha, U_\alpha\}$ gives an differential structure of $M$ immediately follows from prop 3.7.

3 Level set

Given a differentiable function $f$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, we would expect the level set to be a differentiable manifold. But in certain cases, it is not a differentiable manifold. One example would be that $f(x, y) = x^2 + y^2 + 1$, and if we require that $f(x, y) = 0$, we get an empty set. Another example would be $f(x, y) = xy$, if we let $f(x, y) = 0$, we would get $x = 0$ or $y = 0$, which is not a differentiable manifold. The first example is bad because $f$ have no preimage for a given value, the second example is bad because in one of the
preimage point, \( df \) is not surjective in \( R^3 \). It is amazing that it is only two of the bad things that makes the level set not a differentiable manifold.

**Theorem 3.1** Given a differentiable function \( f \) from \( R^n \) to \( R^m \), if \( a \) satisfies that \( f^{-1}(a) \) is not empty, and \( \forall x \in f^{-1}(a) \), \( (df)(x) \) is surjective from \( R^n \) to \( R^m \), then \( f^{-1}(a) \) would be a differentiable manifold.

**Remark 3.1** \( a \) is called the regular value of \( f \).

### 3.1 examples of level set

1. orthogonal group in \( R^n \).

**Proof 3.1** \( A \) is orthogonal iff \( A^TA = I \).

Set \( F(x) = x^T x \), \( F \) is a differentiable map from \( R^n \) to \( R^{n^2+n} \).

We want to prove \( I \) is the regular value of \( F \).

\( I^TI = I \), thus \( F^{-1}(I) \) is not empty.

If \( F(A) = I \), \( \forall C \) symmetric, \( (dF)_A(CA) = C \), thus we get \( (dF)_A \) is surjective.

**Remark 3.2** \( \text{Det}(A) = 1 \) and \( \text{Det}(A) = -1 \) gives two connected componet of the manifold. And in the lee’s book in chapter five.([2])

**Remark 3.3** For a more generalized version, \( A \) lie Matrix group is a differentiable manifold. see [1]

2. sphere \( f(x, y) = x^2 + y^2 + z^2 = 1 \).

**Proof 3.2** \( f(1,0,0) = 1 \).

\( df(x, y, z) = (2x, 2y, 2z) \neq (0, 0, 0), \forall (x, y, z) \) satisfying \( x^2 + y^2 + z^2 = 1 \)

3. torus \( f(x, y, z) = ((x^2 + y^2 - 1)^2 + z^2 = 2 \)

**Proof 3.3** \( f(0, 0, 1) = 2 \)

\( df(x, y, z) = (2x - \frac{2x}{\sqrt{x^2+y^2}}, 2y - \frac{2y}{\sqrt{x^2+y^2}}, 2z) \neq (0, 0, 0) \forall (x, y, z) \) satisfying \( (\sqrt{x^2+y^2} - 1)^2 + z^2 = 2 \).

### 3.1.1 proof of level set of regular value is differentiable manifold

**Proof 3.4** Note that \( df \) is surjective, informally we can say, \( R^n \) lose some freedom in the mapping of \( f \). And our strategy here is adding more freedom to the image so that we can construct a local diffeomorphism between \( R^n \) to constructed manifold. And the diffeomorphism we construct will give a differentiable charts on the level set.

Wlog, \( \left( \frac{\partial f}{\partial x_j} \right)_{1 \leq i \leq m, 1 \leq j \leq m} \) is not singular at \( x \).

Then \( f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n), x_{m+1}, ..., x_n) \), we would have \( (df^*)(x) \) being nonsingular at \( x \).

Then by the inverse function theorem, there is open set \( \Omega \) that diffeomorphic to \( f(x)(\Omega) \). In particular choose \( U_x \) open in \( R^{n-k} \) such that \( (a, U_x) \subset \Omega \). Then \( ((f_x)^{-1}, (a, U_x)) \) gives a parameterization of \( f^{-1}(a) \). Similar as above, we can prove the level set is a regular manifold.
Remark 3.4 If we go back the proof of every regular submanifold of $\mathbb{R}^n$ is a differentiable manifold, we would get every regular submanifold of $\mathbb{R}^n$ is locally a level set.

4 Orientable manifold

An orientable manifold is a differentiable manifold with orientable differentiable charts $(X_\alpha, U_\alpha)$, such that $\det dX_\alpha^{-1}X_\beta$ have positive determinant in the domain of interest. And we say that every orientable differentiable chart give an orientation on $M$.

Theorem 4.1 Every orientable manifold have at least orientations

Proof 4.1 Suppose $X_\alpha, U_\alpha$ is an differentiable charts for $M$, then take $P$ a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, and corresponding Matrix with standard basis has negative determinant. Then we can prove that $(X_\alpha \circ P^{-1}, PU_\alpha)$ give an different orientation on $M$.

Remark 4.1 If the manifold is connected, there exists exactly two orientations. The reader can read chapter 10 in lee’s book.([2])

References
