Examples of differientiable manifold

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1 tangent bundle

Given M a manifold of dimension n with charts $\{(X_\alpha, U_\alpha)\}$, want to construct a differentiable manifold TM for which M is a submanifold. The idea of construction would be attaching every point with its tangent space.
Def 2.6 gives the idea of parameterization.
The coordinate charts is $\{(Y_\alpha, U_\alpha \times R^n)\}$:
$Y_\alpha(x_1, ...x_n, u_1, ..., u_n) = (X_\alpha(x_1, ..., x_n), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i})$.

Proof 1.1 (TM is a differentiable manifold) *$(p, v) \in TM, \exists U_\alpha, X_\alpha$, such that $p \in X_\alpha(U_\alpha)$. Then $(p, v) \in Y_\alpha(U_\alpha, R^n)$, i.e $\{(Y_\alpha, U_\alpha \times R^n)\}$ covers TM.*
Injectivity of Y_α follows from our construction. It remains to show that the charts are compatible. $Y_\alpha(x_1, ..., x_n, u_1, ...u_n) = Y_\beta(x_1^*, ..., x_n^*, u_1^*, ..., u_n^*)$ would imply $(x_1, ..., x_n) = X_\alpha^{-1} X_\beta(x_1^*, ..., x_n^*)$, $(u_1, ..., u_n) = dX_\alpha^{-1} X_\beta(u_1^*, ..., u_n^*)$. Thus,$Y_\alpha^{-1} Y_\beta$ is differentiable from $U_\beta \times \mathbf{R}^n$ to $U_\alpha \times \mathbf{R}^n$.

2 Regular manifold in $\mathbf{R^n}$

Regular manifold M in $\mathbf{R^n}$ is a generalization of regular surface in $\mathbf{R^3}$. Firstly, we are given differientiable maps X_α from $U_\alpha \subset R^m$ to M. More we require $(dX_\alpha)_p$ to be injective from R^m to T_pM. This require that X_α is an immersion from U_α to M,and it require that $m \leq n$.The proof that $\{X_\alpha, U_\alpha\}$ gives an differential structure of M immediately follows from prop 3.7.

3 Level set

Given a differentiable function f from $\mathbf{R^n}$ to $\mathbf{R^m}$, we would expect the level set to be a differentiable manifold. But in certain cases, it is not a differentiable manifold. One example would be that $f(x, y) = x^2+y^2+1$, and if we require that $f(x, y) = 0$,we get an empty set. Another example would be $f(x, y) = xy$, if we let $f(x, y) = 0$, we would get $x = 0$ or $y = 0$, which is not a differentiable manifold. The first example is bad because f have no preimage for a given value, the second example is bad because in one of the preimage point, *df* is not surjective in R^1. It is amazing that it is only two of the bad things that makes the level set not a differientiable manifold.

Theorem 3.1 *Given a differentiable function f from* $\mathbf{R^n}$ *to* $\mathbf{R^m}$*, if a satisfies that* $f^{-1}(a)$ *is not empty, and* $\forall x \in f^{-1}(a)$*,* $(df)(x)$ *is surjective from* $\mathbf{R^n}$ *to* $\mathbf{R^m}$*, then* $f^{-1}(a)$ *would be a differentiable manifold.*

Remark 3.1 *a is called the regular value of f.*

3.1 examples of level set

1. orthogonal group in R^n.

Proof 3.1 *A is orthogonal iff* $A^TA = I$.
Set $F(x) = x^Tx$*, F is a differntiable map from* R^{n^2} *to* $R^{\frac{n^2+n}{2}}$.
We want to prove I is the regular value of F.
$I^TI = I$*, thus* $F^{-1}(I)$ *is not empty.*
If $F(A) = I$*,* $\forall C$ *symmetric,*$(dF)_A(\frac{CA}{2}) = C$*, thus we get* $(dF)_A$ *is surjective.*

Remark 3.2 $Det(A) = 1$ *and* $Det(A) = -1$ *gives two connected componet of the manifold. And in the lee's book in chapter five.([2])*

Remark 3.3 *For a more generalized version, A lie Matrix group is a differentiable manifold.see[1]*

2. sphere $f(x,y) = x^2 + y^2 + z^2 = 1$.

Proof 3.2 $f(1,0,0) = 1$.
$df(x,y,z) = (2x, 2y, 2z) \neq (0,0,0), \forall(x,y,z)$ *satisfying* $x^2 + y^2 + z^2 = 1$

3.torus $f(x,y,z) = (\sqrt{x^2+y^2} - 1)^2 + z^2 = 2$

Proof 3.3 $f(0,0,1) = 2$
$df(x,y,z) = (2x - \frac{2x}{\sqrt{x^2+y^2}}, 2y - \frac{2y}{\sqrt{x^2+y^2}}, 2z) \neq (0,0,0)\forall(x,y,z)$ *satisfying* $(\sqrt{x^2+y^2} - 1)^2 + z^2 = 2$.

3.1.1 proof of level set of regular value is differentiable manifold

Proof 3.4 *Note that df is surjective, informally we can say,* R^n *lose some freedom in the mapping of f. And our strategy here is adding more freedom to the image so that we can construct a local diffeomorphism between* R^n *to constructed manifold. And the diffeomorphism we construct will give a differentiable charts on the level set.*
Wlog, $(\frac{\partial f_i}{\partial x_j})$*(*$1 \leq i \leq m, 1 \leq j \leq m$*) is not singular at x.*
Then $f_x^*(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ...f_m(x_1, ..., x_n), x_{m+1}, ..., x_n)$*, we would have* $(df^*)(x)$ *being nonsingular at x.*
Then by the inverse function theorem, there is open set Ω *that diffeomorphic to* $f_x^*(\Omega)$*.In particular choose* U_x *open in* R^{n-k} *such that* $(a, U_x) \subset \Omega$*. Then* $((f_x^*)^{-1}, (a, U_x))$ *gives a parameterization of* $f^{-1}(a)$*.Similar as above, we can prove the level set is a regular manifold.*

Remark 3.4 *If we go back the proof of every regular submanifold of R^n is a differentiable manifold, we would get every regular submanifold of R^n is locally a level set.*

4 Orientable manifold

An orientable manifold is a differentiable manifold with orientable differentiable charts (X_α, U_α), such that $\det dX_\alpha^{-1} X_\beta$ have positive determinant in the domain of interest.And we say that every orientable differentiable chart give an orientation on M.

Theorem 4.1 *Every orientable manifold have at least orientations*

Proof 4.1 *Suppose X_α, U_α is an differentiable charts for M, then take P a linear transformation from R^n to R^n, and corresponding Matrix with standard basis has negative determinant.Then we can prove that $(X_\alpha \circ P^{-1}, PU_\alpha)$ give an different orientation on M.*

Remark 4.1 *If the manifold is connected, there exists exactly two orientations.The reader can read chapter 10 in lee's book.([2])*

References

[1] Brian C Hall. “Lie Groups, Lie Algebras, and Representations”. In: *Quantum Theory for Mathematicians*. Springer, 2013, pp. 47–59.

[2] John M Lee. “Smooth manifolds”. In: *Introduction to Smooth Manifolds*. Springer, 2013, pp. 1–31.