General definition of tensor.

An \( (M, N) \) tensor is a multilinear function \( f : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R} \) that assigns a real number to \( M \) one-forms and \( N \) vectors into a real number.

Before a deeper discussion, we briefly understand 2 things:

1. \((0, N)\) tensor (because we know vectors).
2. One-forms.

\((0, N)\) tensor:

A tensor of type \((0, N)\) is a function of \( N \) vectors into the real number, which is linear in each of its \( N \) arguments.

Example is metric tensor, curvature tensor, etc.

\[ \langle \mathbf{A}, \mathbf{B} \rangle = G(\mathbf{A}, \mathbf{B}) \]

\[ G(\alpha \mathbf{A} + \beta \mathbf{B}, \mathbf{C}) = \alpha G(\mathbf{A}, \mathbf{C}) + \beta G(\mathbf{B}, \mathbf{C}) \]

A tensor is a geometric object which gives the same real number independently of the reference frame. Hence frame invariant.

\[ G(\mathbf{e}_\alpha, \mathbf{e}_\beta) = G_{\alpha \beta} \]

2. Index in subscript.
A tensor of form $\otimes$ is called covector, a covariant vector or one-form.

They form part of dual vector space.

Hence one-form takes a vector as an argument and gives a real number.

\[
\tilde{s} = \tilde{r} = 0 \rightarrow \tilde{s}(\tilde{v}) = \tilde{r}(\tilde{v}).
\]

Linearity:

\[
\tilde{s} = \tilde{r} = 0 \rightarrow \tilde{s}(\tilde{v}) = \lambda \tilde{r}(\tilde{v}) = \lambda \tilde{r}(\lambda \tilde{v}) = \lambda \tilde{s}(\lambda \tilde{v})
\]

Hence set of all 1-forms satisfy the axioms for a vector space called "dual vector space".

Component of 1-form:

\[
P_\alpha = \tilde{\rho}(\tilde{e}_\alpha).
\]

Compare:

\[
\tilde{\rho}(\tilde{v}) = \tilde{A} = A^\alpha \tilde{e}_\alpha.
\]

$\alpha$-Component of vector:

\[
\tilde{\rho}(\tilde{e}_\alpha).
\]

Hence $P_\alpha$ transforms like $\tilde{e}_\alpha$. Hence called covector.

Action of 1-form on a vector and frame invariance:

\[
\tilde{\rho}(\tilde{A}) = \tilde{\rho}(A^\alpha \tilde{e}_\alpha) = A^\alpha \tilde{\rho}(\tilde{e}_\alpha) = A^\alpha P_\alpha
\]

Frame invariance (also product) not inner product and is scalar.

\[
P_{\tilde{\beta}} = \tilde{\rho}(\tilde{e}_\beta) = \tilde{\rho}(\Lambda_{\tilde{\beta}}^{\alpha} \tilde{e}_\alpha) = \Lambda_{\tilde{\beta}}^{\alpha} P_\alpha
\]

\[
\rightarrow \text{comp. transforms like } \tilde{e}_\beta.
\]
Now consider frame $\tilde{x}$ and $\tilde{y}$

$$\tilde{P}(\tilde{\alpha}) = A^\alpha A^\beta \tilde{p}_\alpha = (\Lambda^\alpha_\beta A^\beta) \Lambda^\gamma_\delta \tilde{p}_\gamma \Lambda^\gamma_\alpha \Lambda^\delta_\beta \tilde{p}_\delta.$$  

By frame invariance,  

$$\tilde{P}(\tilde{\alpha}) = A^\alpha \tilde{p}_\alpha = \Lambda^\alpha_\beta A^\beta \tilde{p}_\beta.$$  

Basis one-form = Called basis dual to $\{\tilde{e}^\alpha\}$.  

$$\tilde{P} = \tilde{p}_\alpha \tilde{e}^\alpha \quad \tilde{P}(\tilde{\alpha}) = \tilde{p}_\alpha \tilde{e}^\alpha(\tilde{\alpha}) = \lambda^\alpha_\beta (A^\beta \tilde{e}_\beta) = \lambda^\alpha_\beta (\tilde{e}_\beta(\tilde{e}_{\beta})).$$

$$\Rightarrow \tilde{e}^\alpha(\tilde{e}_\beta) = \delta^\alpha_\beta.$$  

Transformation  

$$\tilde{e}^\alpha = \Lambda^\alpha_\beta \tilde{e}^\beta,$$  

transforms like a component 1-vector.  

Note: 1-form is tensor of form $(\mathbb{Q})$.

Hence it $\in \mathbb{Q}$.  

There metric tensor is $(\mathbb{Q})$ tensor.

$$\tilde{g}(\tilde{e}^\alpha, \tilde{e}_\beta) = \tilde{g}_{\alpha\beta},$$  

Takes 2 one form
GRADIENT OF A SCALAR ALONG A DIRECTION

\[ \vec{u} = u^\alpha \vec{e}_\alpha \]

\( \tau \): Be the world line (inertial frame time)

It is the parametrization of the world line.

We want to find \( \vec{u} \), which is velocity. Any coordinate system given by metric:

\[ \vec{u} = \frac{d}{d\tau} (t, x, y, z) = \left( \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) \]

We wish to find change of the scalar \( \phi \) along the world line:

\[ \frac{d\phi}{d\tau} = \frac{d\phi}{dt} \frac{dt}{d\tau} + \frac{d\phi}{dx} \frac{dx}{d\tau} + \frac{d\phi}{dy} \frac{dy}{d\tau} + \frac{d\phi}{dz} \frac{dz}{d\tau} \]

\[ \nabla_{\vec{u}} \phi = \nabla \phi \left( \vec{u}, \phi \right) = \phi, \alpha \vec{u}^\alpha \]

Contravariant component of a vector \( \vec{u} \) with one form \( \nabla \phi \).

Hence \( \nabla \phi \) is a 1-form.

\( \nabla \phi (\vec{u}) \) is scalar representing directional derivative of \( \phi \) along \( \vec{u} \).
Hence we can represent
\[ \nabla = \tilde{\omega}^\mu \frac{\partial}{\partial x^\mu} \]

Derivative along \( \bar{u} \)
\[ \nabla_{\bar{u}} = \tilde{\omega}^\mu (u^a \bar{e}_a) \frac{\partial}{\partial x^\mu} = u^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \bar{u}} \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \bar{u}} \]

Tangent vector.
\[ \bar{e}_\mu = \frac{\partial}{\partial x^\mu} \]

COVARIANT DERIVATIVE
OF TENSOR.

Now we consider \((M \times N)\) tensor. Its multilinear in \(M\) one-forms and \(N\) vectors. On component in \(M\) one-forms and \(N\) vectors, say \((3 \times 3)\) tensor
\[ T = T^\alpha_\beta \tilde{\omega}^\lambda \otimes \tilde{\omega}^\mu \otimes \tilde{\omega}^\nu \otimes \bar{e}_\alpha \otimes \bar{e}_\beta \otimes \bar{e}_\gamma \]
\[ \text{takes } 3 \text{ one-forms} \]
\[ \text{takes } 3 \text{ vector} \]
\[ \text{Take } 3 \text{ 1-form} \]

Consider \((1 \times 1)\) tensor
\[ T = T^\alpha_\beta \bar{e}_\alpha \otimes \bar{e}_\beta \]

We wish to find covariant derivative of \(T\).
If $T$ is $(M, N)$-tensor then $\nabla T$ gives $(M + N + 1)$-tensor as there is an additional slot (particular direction) along which the derivative is needed.

If $\phi$ is scalar, $\nabla \phi$ is 1-form or $(0)$ tensor to give $\nabla \phi$ which is again a real number.

Now consider $(1)$ tensor

$$\nabla T = \nabla \left[ T^\alpha_\beta \mathbf{e}^\alpha \otimes \mathbf{w}^\beta \right]$$

will be $(2)$ tensor:

$$= (\nabla T^\alpha_\beta) (\mathbf{e}^\alpha \otimes \mathbf{w}^\beta) + T^\alpha_\beta (\nabla \mathbf{e}^\alpha) \otimes \mathbf{w}^\beta + T^\alpha_\beta \mathbf{e}^\alpha \otimes (\nabla \mathbf{w}^\beta)$$

is a scalar.

Hence $\nabla T^\alpha_\beta$ is 1-form or $(0)$ tensor.

Hence $\nabla T^\alpha_\beta = \left[ \tilde{\nabla} \frac{\partial}{\partial \chi^\gamma} \right] T^\alpha_\beta = \frac{\partial T^\alpha_\beta}{\partial \chi^\gamma} \tilde{\nabla} \gamma \mathbf{w}$

we need a vector or a direction in manifold to get a real no.
Now consider \( \nabla e^a \) is vector or \((1,1)\) tensor as it takes a 1-form to get a number.

Hence \( \nabla e^a \) is a number \((1)\) tensor.

\[
\nabla e^a = \tilde{\omega}^a \frac{\partial e^a}{\partial x^r} = \Gamma^s_{\alpha \beta} e^s \otimes \tilde{\omega}^\alpha .
\]

Similarly

\[
\nabla \tilde{\omega}^\beta = \tilde{\omega}^a \frac{\partial \tilde{\omega}^\beta}{\partial x^r} = -\Gamma^a_{\alpha \nu} (\tilde{\omega}^\alpha \otimes \tilde{\omega}^\nu) .
\]

Can be proven.

Now

\[
\nabla T = T^\alpha_{\beta \gamma} \left( \tilde{\omega}^\gamma \otimes e^\alpha \otimes \tilde{\omega}^\beta \right) + T^\alpha_{\beta \gamma} \left( \Gamma^a_{\alpha \beta} \tilde{\omega}^\alpha \otimes \tilde{\omega}^\gamma \right) e^a \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma
\]

\[
+ T^\alpha_{\beta \gamma} \tilde{e}^\beta \otimes (-\Gamma^a_{\alpha \nu} \tilde{\omega}^\nu \otimes \tilde{\omega}^\alpha)
\]

\[
= (T^\alpha_{\beta \gamma} + T^\alpha_{\beta \gamma} \Gamma^\alpha_{\beta \gamma} - T^\alpha_{\alpha \beta} \Gamma^\alpha_{\beta \gamma}) e^\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma
\]

\( \nabla T \) is \((1,1)\) tensor with \((\nabla T)^\alpha_{\beta \gamma} \) being

\[
(\nabla T)^\alpha_{\beta \gamma} = \Gamma^\alpha_{\beta \gamma} + T^\alpha_{\beta \gamma} \Gamma^\beta_{\alpha \gamma} - T^\alpha_{\alpha \beta} \Gamma^\beta_{\gamma} = T^\alpha_{\beta \gamma}
\]

Called "Covariant Derivative"
Summary

\[ \nabla^ \alpha \nabla_ \beta \Gamma^ \gamma_ {\nu \rho} = \Gamma^ \gamma_ {\nu \rho \beta} - \Gamma^ \gamma_ {\nu \alpha} \Gamma^ \alpha_ {\rho \beta} - \Gamma^ \gamma_ {\nu \lambda} \Gamma^ \lambda_ {\rho \beta} . \]

\[ \Rightarrow \nabla^ \gamma \Gamma^ \gamma_ {\nu \rho} = (\nabla^ \gamma \Gamma)_{\nu \rho} \quad \tilde{\omega}^ \rho_ {\rightarrow} (\tilde{e}^ \rho) \quad \tilde{\omega}^ \nu_ {\rightarrow} (\tilde{e}^ \nu) \quad \tilde{\omega}^ \gamma_ {\rightarrow} (\tilde{e}^ \gamma) . \]

\[ \exists \text{connection}. \]

\[ \nabla^ \gamma A = A^ \gamma_ {\nu \rho} + A^ \gamma_ {\nu} \Gamma^ \rho_ {\rightarrow} + A^ \gamma_ {\lambda} \Gamma^ \rho_ {\rightarrow} . \]

\[ \text{let } \nabla^ \gamma A \text{ be component } \gamma \text{ tensor } A \text{ which } \gamma^{(2)} \text{ tensor} \]

\[ \text{Hence } \quad A^ \gamma_ {\nu \rho} \quad \tilde{\omega}^ \rho_ {\rightarrow} \tilde{e}^ \gamma \quad \tilde{\omega}^ \nu_ {\rightarrow} \tilde{e}^ \gamma . \]

\[ \Rightarrow \nabla^ \gamma A^ \gamma_ {\nu \rho} = A(\tilde{\omega}^ \rho_ {\rightarrow} \tilde{e}^ \gamma , \tilde{\omega}^ \nu_ {\rightarrow} \tilde{e}^ \gamma) . \]

\[ \nabla^ \gamma B^ \gamma_ {\nu \rho} = B^ \gamma_ {\nu \rho} + B^ \lambda_ {\nu} \Gamma^ \gamma_ {\rho \lambda} - B^ \gamma_ {\lambda} \Gamma^ \lambda_ {\nu \rho} . \]
"Spacetime tells matter how to move; matter tells spacetime how to curve." - John Wheeler.

Intrinsic curvature is defined as the difference between an initial vector and the same vector parallel-transported around an infinitesimal loop.

Can be found in chapter 6 "Curved Manifold" in "A First Course in GR".

\[
\frac{\partial}{\partial \lambda} \left[ \frac{\partial}{\partial \lambda} \right] V^\alpha = 0 \quad \text{a} \, \text{d} b \left[ \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\mu} - \Gamma^\mu_{\lambda\nu} \right] V^\mu.
\]

\[
= \left( \delta^\alpha_{\mu} d \delta b \right) R^\lambda_{\mu \lambda}. 
\]

\[
R = R^\alpha_{\mu \lambda} \quad e^\lambda \otimes \tilde{\omega} \otimes \tilde{\omega} \otimes \tilde{\omega}.
\]
\[ R \left( \tilde{w}^s, \tilde{v}, d\alpha e^a, d\beta \tilde{v} \right) \]

A vector \( \tilde{v} \) is moved in an infinite loop.

To get a component of \( \tilde{v} \).

The vector that is parallelly transported.

\[ = R^{\alpha}_{\beta\gamma\delta} \ v^\alpha \tilde{w}^\beta (\tilde{v}) \otimes \tilde{w}^\gamma (d\alpha e^a) \otimes \tilde{w}^\delta (d\beta \tilde{v}) \]

\[ = R^{\alpha}_{\beta\gamma\delta} \ \delta^\alpha_{\delta} d\alpha d\beta \tilde{v}^\gamma = d\tilde{v}^\gamma. \]

**Curvature Tensor** as manifestation of **Gravity**

Consider two nearby geodesics \( V(\lambda) \) and \( V'(\lambda) \) parametrized...
It can be proven:

\[ \nabla \nabla \xi = R(\tilde{\omega}^a, \tilde{\nabla}, \tilde{\nabla}, \xi) \]

or

\[ \nabla \nabla \xi = R(\tilde{\omega}^a, \tilde{\nabla}, \tilde{\nabla}, \xi) = R_{\alpha \beta \gamma} \tilde{\omega}^\alpha \tilde{\omega}^\beta \xi^\gamma \]

Hence the Riemann tensor is the generator of the relative acceleration of nearby geodesics.

Do gravitation is manifest, the relative acceleration of closely spaced freely falling test particles,

**Lowering and Raising the Index**

Consider a vector \( \vec{A} = A^\alpha \vec{e}_\alpha \). We can convert it to a 1-form as

\[ \tilde{A} = g(\nabla, \vec{A}) = A^\alpha g(\nabla, \vec{e}_\alpha) = g_{\alpha \beta} \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta(\vec{A}). \]

This operation involves converting a vector to a 1-form, which requires one additional vector.

\[ \tilde{A} = A^\alpha \tilde{\omega}^\alpha \]

\[ A_\alpha = g_{\alpha \beta} A^\beta \]

\[ \text{Component of } \tilde{A} \text{ 1-form} \]
Similarly

\[ R'_{j'k'\ell} = \Gamma^\alpha_{i'\alpha} R'_{j'k'\ell}. \]

Similarly, \( T^\alpha_{\beta} = g^{i'\alpha} T^\alpha_{\beta} \)

(0) tensor \( \rightarrow \) (1) tensor

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From: Do-Carmo

Definition: A tensor \( T \) of order \( r \) (or basically \( (r) \)-tensor)
on a smooth manifold \( \mathbb{M} \)
a multilinear mapping

\[ T: \mathbb{X}(\mathbb{M}) \times \ldots \times \mathbb{X}(\mathbb{M}) \rightarrow \mathbb{D}(\mathbb{M}) \]

Takes \( r \) vector \quad Gives a scalar

Multilinear implies, if \( X, \ldots, Y_r \in \mathbb{X}(\mathbb{M}), \, X, Y \in \mathbb{X}(\mathbb{M}), \) \( f, g \in \mathbb{D}(\mathbb{M}), \)

\[ T(X, \ldots, fX + gY, \ldots, Y_r) = f \cdot T(X, \ldots, X, \ldots, X) + g \cdot T(X, \ldots, Y, \ldots, Y). \]
Component of tensor.

Tensor $T$ is pointwise object.

Define $\bar{E}_1, \ldots, \bar{E}_n \in \mathfrak{X}(M)$ such that if $q \in U$,

\[
\{ \bar{E}_i(q) \} \text{ form basis of } T_qM.
\]

Hence $\bar{E}_i$ is like a monop frame on $U$.

\[
\Rightarrow \quad Y_r = \bar{Y}_r^i \bar{E}_i \quad \text{Sum unflled over } i
\]

Then $T(Y_1, \ldots, Y_n) = \bar{Y}_r^i \bar{Y}_r^j T(\bar{E}_i, \ldots, \bar{E}_i)$.

\[
\text{Sum on } i
\]

\[
T(\bar{E}_i, \ldots, \bar{E}_i) = \bar{T}_{ij} \ldots \bar{T}_{ij} \quad \text{is component of tensor } T \text{ in } \{ \bar{E}_i \}.
\]

Eg: Curvature tensor.

\[
\bar{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \quad \to \mathcal{O}(M)
\]

Defined by

\[
\bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \langle \bar{R}(\bar{X}, \bar{Y}) \bar{Z}, \bar{W} \rangle.
\]

\[
\text{In fact (0)}
\]

\[
\text{Egvector.}
\]

\[
\bar{R}(\bar{X}_i, \bar{X}_j \bar{X}_k, \bar{X}_l) = \bar{R}_{ij}^k \bar{R}_{kl}^i \quad \text{Component of } \bar{R} \text{ in } \{ \frac{\partial}{\partial \bar{E}_i} \} \text{ frame}.
\]
Eg: Riemannian connection $\nabla$ defined as
\[ \nabla(x, y, z) = \langle \nabla_x y, z \rangle \] is not tensor as $\nabla_x (f y) \neq f \nabla_x y$, so not linear in $y$.

Remark 5.5 hints I-forms and why they are not necessary in Riemannian manifold.

Covariant derivative of tensor

Let $T$ be tensor of order $r$. The covariant differential $\nabla T$ of $T$ is a tensor of order $(r+1)$ given by
\[ \nabla T(x_1, \ldots, x_r, z) = z \left( T(\nabla_z x_1, \ldots, x_r) - T(x_1, \ldots, x_r) \right) - \cdots - T(x_1, \ldots, x_{r-1}, \nabla_z x_r). \]

We need to give the direction along which the variation of tensor is to be observed.

For each $z \in \mathcal{X}(M)$, the covariant derivative $\nabla_z T$ of $T$ relative to $z$ is a tensor of order $r$ given by
\[ \nabla_z T(x_1, \ldots, x_r) = \nabla T(x_1, \ldots, x_r, z). \]
Hence we aim to find change of tangent field along a curve.

\[ \mathbf{\gamma}'(t) = T(t) \mathbf{e}_s(t) \]

\[ T_{PM} = \mathbf{e}^i \text{ basis } T_PM = \mathbf{e}^i(\omega(t)) \]

\[ \{ \mathbf{e}_i(t) \} \text{ basis } \mathcal{T}_x(M) \text{ but is parallelly transported along curve.} \]

Hence \( \nabla_{\mathbf{\gamma}'(t)} \mathbf{e}_i(t) = 0 \).

**Proof**

If tangent field along curve. Hence \( T'(t) \) is function of \( t \):

\[ T(e_1(t), e_2(t), \ldots, e_i(t)) = T_{ij2 \ldots ir} \]

\[ \nabla \mathbf{T} (e_1(t), e_2(t), \ldots, e_i(t)) = \nabla e_i(t) = -T(g_1(t), \ldots, \nabla e_i(t)) \]

\[ = \frac{d}{dt} T_{ij2 \ldots ir} \] change of magnitude.

\[ -T (\nabla e_i(t), \ldots) \]

\[ = \frac{d}{dt} T_{ij2 \ldots ir} \]

In this frame, the component of covariant derivative of \( T \) are usual derivatives of component of \( T \).

Can we say that Christoffel symbol is specifically zero in this frame?
Covariant derivative of metric \( g(x, y) \)

\[
\nabla_x g(x, y) = \mathcal{L}_x \langle x, y \rangle - \langle \nabla_x x, y \rangle - \langle x, \nabla_x y \rangle.
\]

along some curve \( \alpha(t) \)

But for Riemannian metric:

\[
\frac{d}{dt} \langle v, w \rangle = \langle \frac{d}{dt} v, w \rangle + \langle v, \frac{d}{dt} w \rangle.
\]

\[
\Rightarrow \mathcal{L}_x \langle x, y \rangle = \langle \nabla_x x, y \rangle + \langle x, \nabla_x y \rangle.
\]

from (1) and (3)

\[
\nabla_x g(x, y) = 0.
\]