General definition of tensor.

An $\binom{M}{N}$ tensor is multilinear function of M one-form and N vectors into a real number.

Before a deeper discussing we briefly understand 2 things

(1) $\binom{0}{N}$ tensor (Because we know vectors).

(2) One-forms.

$\binom{0}{N}$ tensor.

A tensor of type $\binom{0}{N}$ is a function of N vectors into the real number, which is linear in each of its N arguments.

Example is metric tensor, Curvature tensor. etc.

Let $\langle \vec{A}, \vec{B} \rangle = G(\vec{A}, \vec{B})$

$$G(\alpha\vec{A} + \beta\vec{B}, C) = \alpha G(\vec{A}, \vec{C}) + \beta g(\vec{B}, \vec{C})$$

A tensor is a geometric object which gives the same real number independently of the reference frame. Hence frame invarient.

$$G(\vec{e}_\alpha, \vec{e}_\beta) = \underbrace{G_{\alpha\beta}}_{\text{2 Index in Subscript}}.$$

One-form

A tensor of form $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is called covector, a covariant vector or one-form. They form part of dual vector space.

Hence one-form takes a vector as an argument and gives a real number.

$$\left.\begin{aligned} \tilde{S} &= \tilde{P} + \tilde{q} \\ \tilde{r} &= \alpha \tilde{P} \end{aligned}\right\} \quad \left.\begin{aligned} \tilde{S}(\vec{A}) &= \tilde{P}(\vec{A}) + \tilde{q}(\vec{A}) \\ \tilde{r}(\vec{A}) &= \alpha \tilde{P}(\vec{A}) \end{aligned}\right\} \text{Linearity}$$

Hence set of all 1-forms satisfy the axioms for a vector space called "dual vector space".

Component of 1 form.

$$P_\alpha = \tilde{P}(\vec{e}_\alpha) \xleftrightarrow{\text{Compare.}} \vec{A} = \underbrace{A^\alpha}_{\text{Component of vector}} \vec{e}_\alpha$$

α-Component of $\tilde{P}$ one form.

α is subscript for P_α and $\vec{e}_\alpha$. Hence called

Hence P_α transforms like $\vec{e}_\alpha$. Hence called Covector.

Action of 1 form on a vector. and frame invariance

$$\tilde{P}(\vec{A}) = \tilde{P}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{P}(\vec{e}_\alpha) = \underbrace{A^\alpha P_\alpha}_{\text{Simple contraction}}$$

not inner product and is scalar.

Frame invariance (Also property of tensor).

$$P_{\bar{\beta}} = \tilde{P}(\vec{e}_{\bar{\beta}})$$
$$= \tilde{P}(\underbrace{\Lambda^\alpha_{\bar{\beta}} \vec{e}_\alpha}) = \Lambda^\alpha_{\bar{\beta}} \tilde{P}(\vec{e}_\alpha) = \underbrace{\Lambda^\alpha_{\bar{\beta}} P_\alpha}_{\text{transforms like } \vec{e}_{\bar{\beta}}}.$$

Compare

Now Consider, frame $\bar{\alpha}$ and α

$$\tilde{p}(\vec{A}) = A^{\bar{\alpha}} p_{\bar{\alpha}} = (\Lambda^{\bar{\alpha}}_{\ \beta} A^{\beta})(\Lambda^{\gamma}_{\ \bar{\alpha}} p_{\gamma}).$$

$$= \underbrace{\Lambda^{\gamma}_{\ \bar{\alpha}} \Lambda^{\bar{\alpha}}_{\ \beta}}_{\delta^{\gamma}_{\beta}} A^{\beta} p_{\gamma}.$$

$$\delta^{\gamma}_{\beta} A^{\beta} p_{\gamma}$$

Proves frame invariance. $= A^{\beta} p_{\beta}.$

<u>Basis One-form</u> – Called basis dual to $\{\vec{e}_{\alpha}\}$.

$$\tilde{p} = p_{\alpha} \tilde{\omega}^{\alpha}$$

$$\tilde{p}(\vec{A}) = p_{\alpha} \tilde{\omega}^{\alpha}(\vec{A}) = p_{\alpha} \tilde{\omega}^{\alpha}(A^{\beta} \vec{e}_{\beta}) = p_{\alpha} A^{\beta} (\tilde{\omega}^{\alpha}(\vec{e}_{\beta}))$$

$$= p_{\alpha} A^{\alpha}$$

$$\boxed{\Rightarrow \tilde{\omega}^{\alpha}(\vec{e}_{\beta}) = \delta^{\alpha}_{\beta}.}$$

Transformation

$$\boxed{\tilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\ \beta} \tilde{\omega}^{\beta}.}$$ transforms like a component of vector

<u>Note</u>: 1 form is tensor of form $\binom{0}{1}$. → They take a vector.

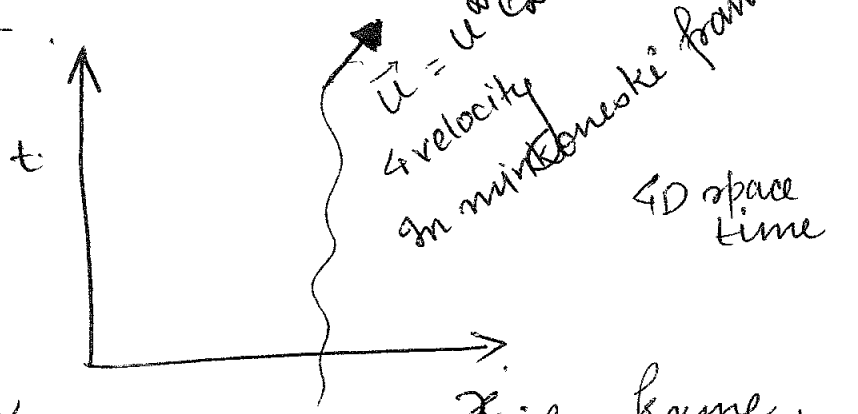
~~Hence its~~

<u>Inverse metric tensor</u> is $\binom{2}{0}$ tensor.

$$\tilde{G}(\underbrace{\tilde{\omega}^{\alpha}, \tilde{\omega}^{\beta}}_{\text{Takes 2 one form}}) = \tilde{G}^{\alpha\beta}.$$

GRADIENT OF A SCALAR (OR DIRECTIONAL DER. of FUNCTION) ALONG A DIRECTION.

$$\vec{u} = u^{\alpha} \vec{e}_{\alpha}$$



τ : Be the real line (inertial frame time)

It is the parametrization of the world line.

In local inertial frame (Tangent space) it is tangent to the geodesic.

We want to ~~take~~ find $\vec{u}$ which is velocity.

$$\vec{u} = \frac{d}{d\tau}\underbrace{(t, X, Y, Z)}_{\text{any Coordinate system given by metric}} = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$$

We wish to find change of the scalar ϕ along the world line.

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau}$$

$$\nabla_{\vec{u}}\phi = \nabla\phi(\vec{u}) = \phi_{,\alpha} u^{\alpha}$$

Contraction of vector $\vec{u}$ with one-form $\nabla\phi$.

argument is a vector

Hence $\nabla\phi$ is a 1 form.

$\nabla\phi(\vec{u})$ is scalar representing directional derivative of ϕ along $\vec{u}$.

Hence we can represent

$$\nabla = \tilde{\omega}^{\mu} \frac{\partial}{\partial x^{\mu}}$$

Derivative along $\vec{u}$

$$\nabla_{\vec{u}} = \tilde{\omega}^{\mu}\left(u^{\alpha}\vec{e}_{\alpha}\right)\frac{\partial}{\partial x^{\mu}} = u^{\mu}\underbrace{\frac{\partial}{\partial x^{\mu}}}_{\text{Tangent vector.}} = \frac{\partial x^{\mu}}{\partial \tau}\frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial \tau}$$

$$\vec{e}_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}.$$

COVARIENT DERIVATIVE OF TENSOR.

Now we consider $\binom{M}{N}$ tensor. Its multilinear in M one-forms and N-vectors. In component form we represent, say $\binom{3}{3}$ tensor

$$T = T^{\overbrace{\alpha\beta\nu}^{\text{takes 3 one form}}}_{\underbrace{\lambda\mu\delta}_{\text{Takes 3 vector}}} \underbrace{\tilde{\omega}^{\lambda}\otimes\tilde{\omega}^{\mu}\otimes\tilde{\omega}^{\delta}}_{\text{takes 3 vector}} \otimes \underbrace{\vec{e}_{\alpha}\otimes\vec{e}_{\beta}\otimes\vec{e}_{\nu}}_{\text{Take 3 1-form}}$$

Consider $\binom{1}{1}$ tensor

$$T = T^{\alpha}_{\beta}\,\vec{e}_{\alpha}\otimes\tilde{\omega}^{\beta}.$$

We wish to find Co-variant derivative of T.

If T is $\binom{M}{N}$ tensor then ∇T gives $\binom{M}{N+1}$ tensor as there is an additional slot (particular direction) along which the derivative is needed)

Eg $\underbrace{\phi \text{ is scalar}}_{\binom{0}{0}\text{ tensor}}$, $\nabla\phi$ is 1 form or $\binom{0}{1}$, $\vec{\nabla}\phi$ consume $\vec{u}$ to give $\nabla_{\vec{u}}\phi$ which is again a real number.

<u>Now Consider $\binom{1}{1}$ tensor</u>

$$\underbrace{\nabla T}_{\text{will be } \binom{1}{2} \text{ tensor.}} = \nabla\left[T^{\alpha}_{\beta}\,\vec{e}_{\alpha}\otimes\tilde{\omega}^{\beta}\right]$$

$$= \underbrace{\left(\nabla T^{\alpha}_{\beta}\right)}_{\text{is a scalar}}\left(\vec{e}_{\alpha}\otimes\tilde{\omega}^{\beta}\right) + T^{\alpha}_{\beta}\left(\nabla\vec{e}_{\alpha}\right)\otimes\tilde{\omega}^{\beta} + T^{\alpha}_{\beta}\,\vec{e}_{\alpha}\otimes\left(\nabla\tilde{\omega}^{\beta}\right)$$

Hence $\nabla T^{\alpha}_{\beta}$ is 1 form or $\binom{1}{0}$ tensor.

Hence $\nabla T^{\alpha}_{\beta} = \left[\tilde{\omega}^{\gamma}\frac{\partial}{\partial x^{\gamma}}\right]T^{\alpha}_{\beta} = \frac{\partial T^{\alpha}_{\beta}}{\partial x^{\gamma}}\tilde{\omega}^{\gamma} = \underbrace{T^{\alpha}_{\beta,\gamma}\,\tilde{\omega}^{\gamma}}_{\text{we need a vector or a direction in manifold to get a real no.}}$

Now Consider

$\nabla \vec{e}_\alpha$ is vector or $\binom{1}{0}$ tensor as it takes a 1 form to get a number.

Hence $\nabla \vec{e}_\alpha$ is a ~~number~~ $\binom{1}{1}$ tensor.

$$\nabla \vec{e}_\alpha = \tilde{\omega}^\gamma \underbrace{\frac{\partial \vec{e}_\alpha}{\partial x^\gamma}}_{\Gamma^\rho_{\alpha\beta} \vec{e}_\rho} = \Gamma^\rho_{\alpha\beta} \vec{e}_\rho \otimes \tilde{\omega}^\gamma.$$

Similarly $\underbrace{\nabla \tilde{\omega}^\beta = \tilde{\omega}^\gamma \frac{\partial \tilde{\omega}^\beta}{\partial x^\gamma} = -\Gamma^\alpha_{\alpha\gamma} (\tilde{\omega}^\alpha \otimes \tilde{\omega}^\gamma)}_{\text{Can be proved.}}$

Now

$$\nabla T = T^\alpha_{\beta,\gamma} (\tilde{\omega}^\gamma \otimes \vec{e}_\alpha \otimes \tilde{\omega}^\beta) + T^\alpha_\beta (\Gamma^a_{\alpha\gamma} \vec{e}_a \otimes \tilde{\omega}^\gamma) \otimes \tilde{\omega}^\beta + T^\alpha_\beta \vec{e}_\alpha \otimes (-\Gamma^\alpha_{a\gamma} \tilde{\omega}^a \otimes \tilde{\omega}^\gamma)$$

$$= \underbrace{(T^\alpha_{\beta,\gamma} + T^a_\beta \Gamma^\alpha_{a\gamma} - T^\alpha_a \Gamma^a_{\beta\gamma})}_{\nabla T \text{ is } \binom{1}{2} \text{ tensor with } (\nabla T)^\alpha_{\beta\gamma} \text{ component}} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma.$$

being $\boxed{(\nabla T)^\alpha_{\beta\gamma} = T^\alpha_{\beta,\gamma} + T^a_\beta \Gamma^\alpha_{a\gamma} - T^\alpha_a \Gamma^a_{\beta\gamma} = T^\alpha_{\beta;\gamma}}$

Called "Comma" goes to "Semicolon"

Summary

(1) $$\nabla_\beta T_{\mu\nu} = T_{\mu\nu,\beta} - T_{\alpha\nu}\Gamma^{\alpha}_{\mu\beta} - T_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}.$$

$$\Rightarrow \nabla_\beta T_{\mu\nu} = (\nabla T)_{\beta\mu\nu}\ \tilde{\omega}^{\beta}(\vec{e}_\beta)\otimes\tilde{\omega}^{\mu}(\vec{e}_\mu)\,\tilde{\omega}^{\nu}(\vec{e}_\nu)$$

(2) $$\nabla_\beta A^{\mu\nu} = A^{\mu\nu}_{\ \ ,\beta} + A^{\alpha\nu}\Gamma^{\mu}_{\alpha\beta} + A^{\mu\alpha}\Gamma^{\nu}_{\alpha\beta}.$$

Let $\nabla_\beta A^{\mu\nu}$ be component of tensor A which is $\binom{2}{1}$ tensor

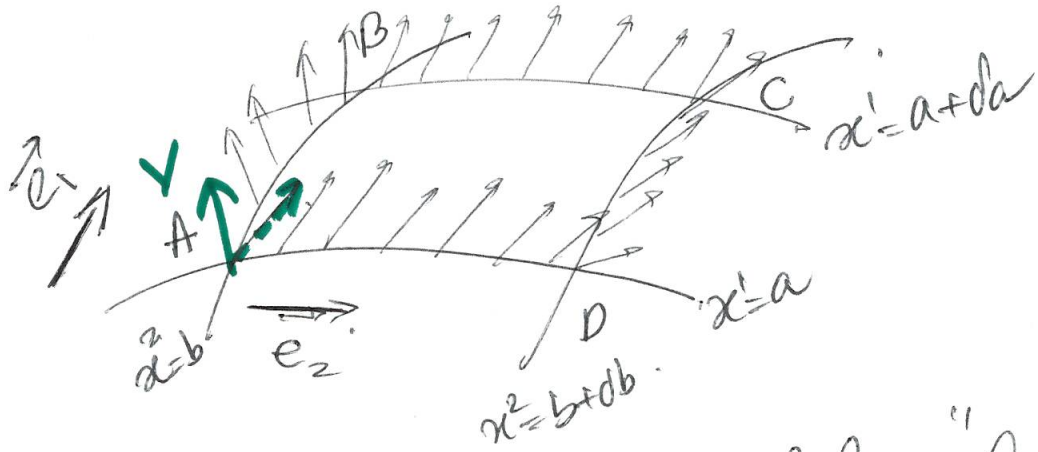
Hence
$$A = \nabla_\beta A^{\mu\nu}\ \tilde{\omega}^{\beta}\otimes\vec{e}_\mu\otimes\vec{e}_\nu$$
$$\Rightarrow \nabla_\beta A^{\mu\nu} = A(\tilde{\omega}^{\beta}, \vec{e}_\mu, \vec{e}_\nu)$$

(3) $$\nabla_\beta B^{\mu}_{\nu} = B^{\mu}_{\nu,\beta} + B^{\alpha}_{\nu}\Gamma^{\mu}_{\alpha\beta} - B^{\mu}_{\alpha}\Gamma^{\alpha}_{\nu\beta}.$$

PHYSICAL INTERPRETATION OF CURVATURE AS MENIFESTATION OF GRAVITATION.

"Spacetime tells matter how to move; matter tells spacetime how to curve". John Wheeler.

Intrinsic curvature is defined as the difference between an initial vector and the same vector parallel-transported around an infinitesimal loop.



Can be proved (Refer chapter 6 "Curved Manifold" in "A first course in GR".

$$dV^\alpha = da\,db\left[\Gamma^\alpha_{\gamma\sigma,\lambda} - \Gamma^\alpha_{\gamma\lambda,\sigma} + \Gamma^\alpha_{\nu\lambda}\Gamma^\nu_{\gamma\sigma} - \Gamma^\alpha_{\nu\sigma}\Gamma^\nu_{\gamma\lambda}\right]V^\gamma.$$

$$= (da\,db\,V^\gamma)\; R^\alpha_{\gamma\lambda\sigma}.$$

$$\boxed{R = R^\alpha_{\gamma\lambda\sigma}\;\vec{e}_\alpha \otimes \tilde{\omega}^\gamma \otimes \tilde{\omega}^\lambda \otimes \tilde{\omega}^\sigma.}$$

$$R(\tilde{\omega}^{\rho}, \vec{V}, \underbrace{da\,\vec{e}_a, db\,\vec{e}_\lambda}_{\text{2 vector of this infinite loop.}})$$

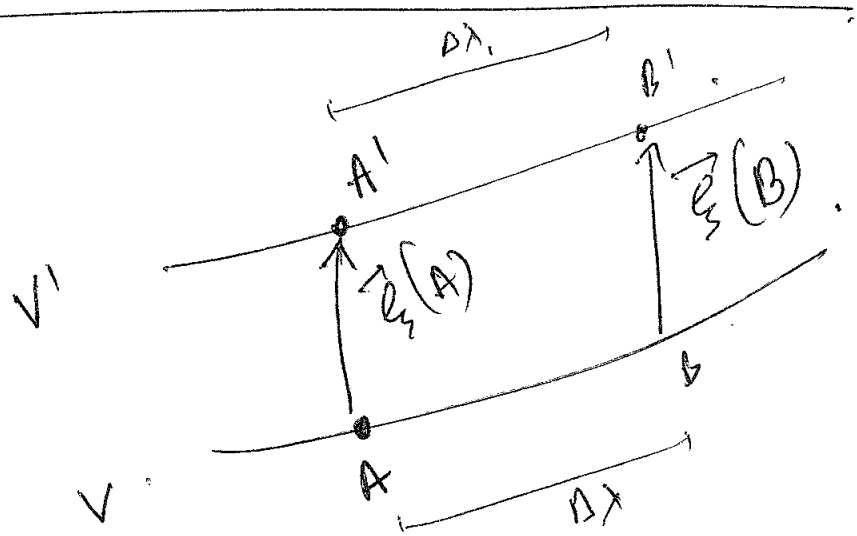
To get ρ component of $\delta\vec{V}$

The vector that is parallely transported.

$$= R^{\alpha}_{\beta\gamma\nu}\, \vec{e}_\alpha(\tilde{\omega}^{\rho}) \otimes \tilde{\omega}^{\beta}(\vec{V}) \otimes \tilde{\omega}^{\gamma}(db\,\vec{e}_\lambda) \otimes \tilde{\omega}^{\nu}(da\,e_a)$$

$$= R^{\rho}_{\gamma\lambda a}\, \delta a\, \delta b\, V^{\gamma} = \delta V^{\rho}.$$

CURVATURE TENSOR AS MENIFESTATION OF GRAVITY



Consider 2 nearby Geodesic $V(\lambda)$ and $V'(\lambda)$.
$\lambda \to$ parameter.

It can be proved

$$\nabla\nabla\vec{\xi} = R(\tilde{w}^{\alpha}, \vec{V}, \vec{V}, \vec{\xi})$$

or

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\vec{\xi} = R(\tilde{w}^{\alpha}, \vec{V}, \vec{V}, \vec{\xi}) = R^{\alpha}{}_{\gamma\gamma\beta}(V^{\gamma}, V^{\gamma}, \xi^{\beta})$$

Hence Reimann tensor is the generator of the relative acceleration of nearby geodesic

Or gravitation is manifest in the relative acceleration of closely spaced freely falling test particle.

LOWERING AND RAISING THE INDEX

Consider a vector $\vec{A} = A^{\alpha}\vec{e}_{\alpha}$. we can convert it to 1 form as

$$\tilde{A} = \underbrace{g(\ \ , \underbrace{\vec{A}}_{\text{vector}})}_{\text{needs 1 vector hence 1-form.}} = A^{\alpha} g(\ \ , \vec{e}_{\alpha}) = g_{\alpha\beta}\, \tilde{w}^{\alpha} \otimes \tilde{w}^{\beta}(\vec{A})$$

$$= g_{\alpha\beta} A^{\alpha}\, \tilde{w}^{\alpha} \otimes \tilde{w}^{\beta}(\vec{e}_{\alpha})$$

$$= g_{\alpha\beta}\, \delta^{\beta}_{\alpha}\, A^{\alpha}\, \tilde{w}^{\alpha}$$

$$= \underbrace{(g_{\alpha\beta} A^{\beta})}_{\alpha \text{ component of } \tilde{A} \text{ 1-form}} \tilde{w}^{\alpha}$$

$$\tilde{A} = A_{\alpha}\tilde{w}^{\alpha} \qquad \boxed{A_{\alpha} = g_{\alpha\beta}A^{\beta}.}$$

Similarly

$$R_{ijkl} = g_{i\alpha} R^{\alpha}_{jkl}.$$

Similarly $T_{\gamma\beta} = g_{\gamma\alpha} T^{\alpha}_{\beta}$.

$\binom{0}{2}$ tensor $\quad$ $\binom{1}{1}$ tensor

FROM. DO-CARMO.

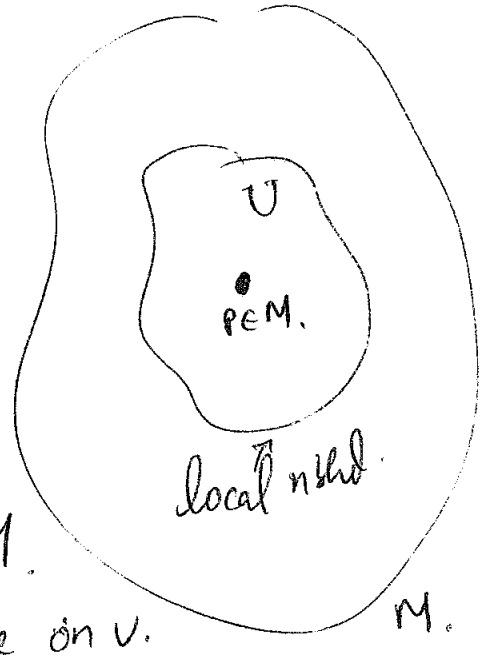
<u>Definition 5.1</u>) A tensor T of order 'r' (or basically $\binom{0}{r}$ tensor) on a smooth manifold is a multilinear mapping

$$T : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{\text{Takes } r \text{ vector}} \longrightarrow \underbrace{D(M)}_{\text{Gives a Scalar}}$$

Multilinear implies, if $Y_1, \ldots, Y_r \in \mathcal{X}(M)$, $X, Y \in \mathcal{X}(M)$, $f, g \in D(M)$

$$T(Y_1, \ldots, fX + gY, \ldots, Y_r) = f\,T(Y_1, \ldots, X, \ldots, Y_r) + g\,T(Y_1, \ldots, Y, \ldots, Y_r)$$

Component of tensor.



Tensor T is pointwise object

Define $\vec{E}_1, \dots, \vec{E}_n \in \mathcal{X}(M)$
such that if $q \in U$.

$\{\vec{E}_i(q)\}$ form basis of T_qM.

Hence $\vec{E}_i$ is like a moving frame on U.

$\Rightarrow Y_r = y_{i_r} E_{i_r}$ Sum implied over i

then $T(Y_1, \dots, Y_r) = \underbrace{y_{i_1} \cdots y_{i_r}}_{\text{sum on } i} T(E_{i_1}, \dots, E_{i_r})$.

$T(E_{i_1}, \dots, E_{i_r}) = T_{i_1, \dots, i_r}$ is component of tensor T in $\{E_i\}$.

Eg: Curvature tensor.

$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \longrightarrow \mathcal{D}(M)$.

Defined by

$R(X, Y, Z, W) = \langle R(X,Y)Z, W \rangle$. Infact $\binom{0}{4}$ tensor.

$R(X_i, X_j, X_k, X_l) = R_{ijkl}$ is component of R in $\left\{\frac{\partial}{\partial x_i}\right\}$ frame.

Eg: Riemannian Connection ∇ defined as

$$\nabla(X, Y, Z) = \langle \nabla_X Y, Z \rangle \quad \text{is not tensor}$$

as $\nabla_X (fY) \neq f \nabla_X Y$ as not linear in Y.

Remark 5.5 hints 1-forms and why they are not necessary in Riemmanian manifold.

Covariant derivative of tensor

Let T be tensor of order r. The Covariant differential ∇_T of T is a tensor of order $(r+1)$ given by

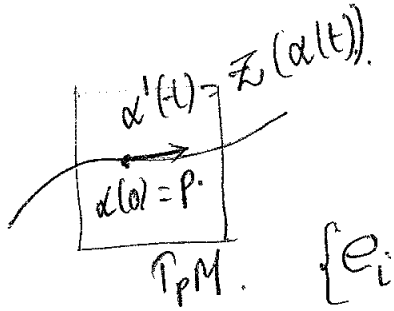
$$\nabla_T(Y_1, \ldots, Y_r, Z) = \overbrace{Z\left(T(Y_1, \ldots, Y_r)\right)}^{\text{related to change of magnitude of component of tensor}} - T(\nabla_Z Y_1 \ldots Y_r) - \cdots - T(Y_1, \ldots, Y_{r-1}, \nabla_Z Y_r).$$

need to give the direction along which the variation of tensor is to be observed!

For each $Z \in \mathfrak{X}(M)$, the Covariant derivative $\nabla_Z T$ of T relative to Z is a tensor of order r given by

$$\nabla_Z T(Y_1, \ldots, Y_r) = \nabla_T(Y_1 \ldots Y_r, Z).$$

Hence we aim to find change of tangent field along tangent curve.



$\{e_i\}$ Basis $T_pM = T_{\alpha(0)}M$.

$\{e_i(t)\}$ Basis $T_{\alpha(t)}M$ but is parallely transported along curve.

Hence $\nabla_{\alpha'(t)} \vec{e_i}(t) = 0$.

~~P(e~~

T is tensor field along curve. Hence $T(t)$ is function of t:

$$T(e_{i_1}(t), e_{i_2}(t), \ldots, e_{i_r}(t)) = T_{i_1 i_2 \cdots i_r}$$

$$\nabla_Z T(e_{i_1}(t), e_{i_2}(t), \ldots, e_{i_r}(t)) = \underbrace{\frac{d}{dt} T_{i_1 i_2 \cdots i_r}}_{\text{change of magnitude.}}$$

$$- T(\nabla_Z e_{i_1}(t), \ldots) - \cdots - T(e_{i_1}(t), \ldots, \nabla_Z e_{i_r}(t))$$

$$\Rightarrow \boxed{\nabla_Z T(e_{i_1}(t), e_{i_2}(t), \ldots, e_{i_r}(t)) = \frac{d}{dt} T_{i_1 i_2 i_3 \cdots i_r}}$$

In this frame, the component of derivative of T are usual derivatives of component of T.

Can we say that Christoffel symbol is specifically zero in this frame?

Covariant derivative of metric $G(X,Y)$

$$\nabla_Z G(X,Y) = \underbrace{Z\langle X,Y\rangle}_{\frac{d}{dt}\langle v,w\rangle \text{ along some curve } \alpha(t)} - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \quad (1)$$

But for riemannian metric

$$\frac{d}{dt}\langle v,w\rangle = \left\langle \frac{Dv}{dt}, w\right\rangle + \left\langle v, \frac{Dw}{dt}\right\rangle.$$

$$\text{so } Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \quad (2)$$

from (1) and (2)

$$\nabla_Z G(X,Y) = 0.$$