$$\frac{d}{dt}\left(2\left\langle \frac{DJ}{dt}, J(t)\right\rangle\right) =$$

$$2\left\langle \frac{D^2J}{dt^2}, J(t)\right\rangle + 2\left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle$$

evaluate at 0:

$$= 2 \left|\frac{DJ}{dt}(0)\right|^2$$

$$J(t) = d(\exp_q)_{tv}(tw) = \frac{\partial f}{\partial s}(t,0)$$

$$\frac{DJ}{dt} = \frac{D}{dt}\left(d(\exp_q)_{tv}(tw)\right)$$

$$= \frac{D}{dt}\left(t\, d(\exp_q)_{tv}(w)\right)$$

$$= d(\exp_q)_{tv}(w) + t\frac{D}{dt}\left(d(\exp_q)_{tv}(w)\right)$$

evaluate at $t=0$:

$$\frac{DJ}{dt}(0) = d(\exp_q)_0(w) + 0 = w.$$

So $\frac{d}{dt}|J(t)|^2\Big|_{t=0} = 2|w|^2 = 2.$

<u>Next</u>: $\frac{d^3}{dt^3}\langle J(t), J(t)\rangle$

$$= \frac{d}{dt}\left(2\left\langle \frac{D^2J}{dt^2}, J(t)\right\rangle + 2\left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle\right)$$

$$= 2\left\langle \frac{D^3J}{dt^3}, J(t)\right\rangle + 6\left\langle \frac{D^2J}{dt^2}, \frac{DJ}{dt}\right\rangle$$

<u>at 0</u>: $$= 6\left\langle \frac{D^2J}{dt^2}(0), \frac{DJ}{dt}(0)\right\rangle$$

Jacobi equation: $$\frac{D^2J}{dt^2} + R(\gamma', J)\gamma' = 0$$

evaluate at 0: $$\frac{D^2J}{dt^2}(0) = 0$$

So $$\frac{d^3}{dt^3}|J(t)|^2\Big|_{t=0} = 0$$

Next and last derivative:

$$\frac{d^4}{dt^4}|J(t)|^2 = \frac{d}{dt}\left(2\left\langle \frac{D^3J}{dt^3}, J\right\rangle + 6\left\langle \frac{D^2J}{dt^2}, \frac{DJ}{dt}\right\rangle\right)$$

$$= 2\left\langle \frac{D^4J}{dt^4}, J\right\rangle + 8\left\langle \frac{D^3J}{dt^3}, \frac{DJ}{dt}\right\rangle + 6\left\langle \frac{D^2J}{dt^2}, \frac{D^2J}{dt^2}\right\rangle$$

at 0 $$= 8\left\langle \frac{D^3J}{dt^3}, \frac{DJ}{dt}\right\rangle(0)$$

differentiate the Jacobi equation:

$$\frac{D^3J}{dt^3} + R\left(\gamma', \frac{DJ}{dt}\right)\gamma' = 0.$$

So $$\frac{D^3J}{dt^3}(0) = -R(v, w)v$$

the coefficient of the Taylor expan. is: $\frac{8!}{4!}\langle -R(v,w)v, w\rangle$

$$= -\frac{1}{3} K(\sigma) \qquad \square.$$

<u>Remark:</u> γ' & $t\gamma'$ are Jacobi fields

$$\frac{D^2\gamma'}{dt^2} = 0 = R(\gamma', \gamma')\gamma'$$

So we restrict ourselves to Jacobi fields normal to γ.

<u>Example:</u> Manifolds with constant sectional curvature.

<u>Recall:</u> $K(\sigma) = K_0 \quad \forall \sigma. \underset{(3.4)}{\equiv}$

$$(X,Y,Z,T) = -K_0\left(\langle X,T\rangle\langle Y,Z\rangle - \langle Y,T\rangle\langle X,Z\rangle\right)$$

where $(X,Y,Z,T) = \langle R(X,Y)Z, T\rangle$

So $\langle R(\gamma', J)\gamma', T\rangle =$

$$= -K_0\left(\langle \gamma', T\rangle\langle \gamma', J\rangle - \langle J, T\rangle\langle \gamma', \gamma'\rangle\right)$$

$\forall$ vector field T.

So $R(\gamma', J)\gamma' = -K_0(\langle\gamma', J\rangle\gamma' - \langle\gamma', \gamma'\rangle J)$

If J is normal to γ, and γ' has length 1, we obtain:

$$R(\gamma', J)\gamma' = + K_0 J$$

and the Jacobi equation becomes

$$\frac{D^2 J}{dt^2} + K_0 J = 0$$

Recall that if $\{e_1(t), \dots, e_n(t)\}$ is an orthonormal frame and if we write $J = \sum_{i=1}^{n} g_i e_i$, then the Jacobi equation becomes:

$$\forall j \qquad g_j''(t) + \sum_{i=1}^{n} a_{ij} g_i(t) = 0$$

where
$$\begin{aligned} a_{ij} &= \langle R(\gamma', e_i)\gamma', e_j\rangle \\ &= K_0(-\langle\gamma', e_j\rangle\langle\gamma', e_i\rangle + \langle\gamma', \gamma'\rangle\langle e_i, e_j\rangle) \\ &= K_0(\delta_{ij} - \langle\gamma', e_j\rangle\langle\gamma', e_i\rangle) \end{aligned}$$

Choose $e_1 = \gamma'$, then $a_{ij} = K_0(\delta_{ij} - \delta_{1i}\delta_{1j})$

and $g_1 = 0$

If, for instance $J = g_2 e_2$, then

$$g_2'' + a_{22} g_2 = 0$$

and $a_{22} = K_0$

so $g_2'' + K_0 g_2 = 0$

$g_2 =$ combination of sine and cosine

<u>Proposition</u>: (2.4) $\gamma : [0,a] \to M$ a geodesic, $\gamma(0) = q$.

J a Jacobi field along γ s.t. $\underline{J(0) = 0}$.

Denote $\gamma'(0) = v$, $\frac{DJ}{dt}(0) = w$.

Then $J(t) = d(\exp_q)_{tv}(tw)$

$$= \frac{\partial f}{\partial s}(t,0) \text{ where}$$

$$f(t,s) = \exp_q\left(\frac{t}{a} v(s)\right)$$

with $v(s)$ a curve in $T_{av}(T_q M)$

with $v(0) = av$ and $v'(0) = w$.

<u>Proof</u>: Both vector fields satisfy the Jacobi equation, so they are equal

if and only if they satisfy the same initial conditions.

$$J(0) = 0 \quad \text{and} \quad d(\exp_q)_{tv}(0) = 0$$

$$\frac{DJ}{dt}(0) = w \text{ , we saw earlier that}$$

$$\frac{D}{dt}\left(d(\exp_q)_{tv}(tw)\right) = w.$$

$\square$.

Conjugate points and the critical point of exp.

A critical point of $\exp_q$ $(q \in M)$ is a point where $d(\exp_q)$ fails to be surjective $\Leftrightarrow$ $d(\exp_q)$ fails to be injective because T_qM and M have the same dim.

$d(\exp_q)_v$ fails to be injective if $\exists\, w \in T_v(T_qM)$ s.t.

$$d(\exp_q)_v(w) = 0.$$

Def: Let $p \in M$ be a point, a point $q \in M$ is a conjugate point along a geodesic γ if

$$\gamma : [0, a] \longrightarrow M.$$

$\gamma(0) = p$, $\exists$ t_0 s.t. $\gamma(t_0) = q$ and $\exists$ Jacobi field along γ s.t.

$$J(0) = 0 = J(t_0)$$

Note: then $J(t) = d(\exp_p)_{tv}(tw)$ where $v = \gamma'(0)$ and $w = \frac{DJ}{dt}(0)$

and $J(0) = 0 = d(\exp_p)_{t_0 v}(t_0 w)$.

So $t_0 v$ is a critical point of $(\exp_p)$ ↑ not needed

Def: The multiplicity of a conjugate point is the number of linearly independent Jacobi fields with the property $J(0) = J(t_0) = 0$.

Note that γ' and $t\gamma'$ are Jacobi fields. $\gamma'(0) \neq 0$ but $(t\gamma')(0) = 0$

$$\frac{D}{dt}(t\gamma') = \gamma' + t\frac{D\gamma'}{dt} = \gamma' \text{ never } 0.$$

There are ~~at most~~ n linearly indep. Jacobi fields s.t. $J(0)=0$ one of them is never 0 outside of p, then $\exists$ at most $n-1$ linearly indep. Ji fields that can vanish at $\gamma(t_0)$. So multiplicity $\leq n-1$.