\[
\frac{d}{dt} \left( 2 \left< \frac{DJ}{dt}, J(t) \right> \right) =
2 \left< \frac{D^2 J}{dt^2}, J(t) \right> + 2 \left< \frac{DJ}{dt}, \frac{DJ}{dt} \right>
\]
evaluate \text{ at } t = 0:
\[
= 2 \left| \frac{DJ}{dt} (0) \right|^2
\]

\[
J(t) = d \left( \exp_\theta \right)_{\text{tr}} (tw) = \frac{df}{ds} (t, 0)
\]

\[
\frac{DJ}{dt} = \frac{D}{dt} \left( d \left( \exp_\theta \right)_{\text{tr}} (tw) \right)
\]

\[
= \frac{D}{dt} \left( t \ d \left( \exp_\theta \right)_{\text{tr}} (w) \right)
\]

\[
= d \left( \exp_\theta \right)_{\text{tr}} (w) + t \frac{D}{dt} \left( d \left( \exp_\theta \right)_{\text{tr}} (w) \right)
\]
evaluate \text{ at } t = 0:
\[
\frac{DJ}{dt} (0) = d \left( \exp_\theta \right)_{\text{tr}} (w) + 0 = w.
\]

So, \[
\frac{d}{dt} \left| J(t) \right|^2 \bigg|_{t=0} = 2 |w|^2 = 2.
\]

Next:
\[
\frac{d^2}{dt^2} \left< J(t), J(t) \right>
= \frac{d}{dt} \left( 2 \left< \frac{D^2 J}{dt^2}, J(t) \right> + 2 \left< \frac{DJ}{dt}, \frac{DJ}{dt} \right> \right)
\]
\[ = 2 \left< \frac{D^3 J}{dt^3}, J(t) \right> + 6 \left< \frac{D^2 J}{dt^2}, \frac{DJ}{dt} \right> \]

\[ \text{at } t=0 : \quad = 6 \left< \frac{D^2 J}{dt^2}(0), \frac{DJ}{dt}(0) \right> \]

Jacobi equation:
\[ \frac{D^2 J}{dt^2} + R(\theta', J) \theta' = 0 \]

evaluate at t=0:
\[ \frac{D^2 J}{dt^2}(0) = 0 \]

So:
\[ \frac{d^3}{dt^3} |J(t)|^2 \bigg|_{t=0} = 0 \]

Next and last derivative:
\[ \frac{d^4}{dt^4} |J(t)|^2 = \frac{d}{dt} \left( 2 \left< \frac{D^2 J}{dt^2}, J \right> + 6 \left< \frac{D J}{dt^2}, \frac{DJ}{dt} \right> \right) \]
\[ = 2 \left< \frac{D^4 J}{dt^4}, J \right> + 8 \left< \frac{D^3 J}{dt^3}, \frac{DJ}{dt} \right> + 6 \left< \frac{D^2 J}{dt^2} \frac{DJ}{dt^2} \right> \]

\[ \text{at } t=0 = 8 \left< \frac{D^3 J}{dt^3}, \frac{DJ}{dt} \right>(0) \]

differentiate the Jacobi equation:
\[ \frac{D^3 J}{dt^3} + R(\theta', \frac{DJ}{dt}) \theta' = 0 \]

So:
\[ \frac{D^3 J}{dt^3}(0) = -R(\nu, w) \nu \]
the coefficient of the Taylor expansion is:
\[
\frac{8}{4!} \left\langle -R(v,w)v, w \right\rangle
\]
\[
= -\frac{1}{3} K(\sigma)
\]  \(\square\).

Remark: \(\gamma' \neq t \phi'\) are Jacobi fields.

\[
\frac{D^2 \phi'}{dt^2} = 0 = R(\gamma', \gamma) \gamma'
\]

So we restrict ourselves to Jacobi fields normal to \(\sigma\).

Example: Manifolds with constant sectional curvature.

Recall: \(K(\sigma) = K_0 \forall \sigma \in \mathbb{S}\) \(\Rightarrow (3.4)\)

\[
(X,Y,Z,T) = K_0 \left( \left\langle X,T \right\rangle \left\langle Y,Z \right\rangle - \left\langle Y,T \right\rangle \left\langle X,Z \right\rangle \right)
\]

where \((X,Y,Z,T) = \left\langle R(X,Y)Z, T \right\rangle\)

So:
\[
\left\langle R(\phi', J)\phi', T \right\rangle = -K_0 \left( \left\langle \phi', T \right\rangle \left\langle \phi', J \right\rangle - \left\langle J, T \right\rangle \left\langle \phi', \phi' \right\rangle \right)
\]

\(\forall\) vector field \(T\).
So \( R(\xi', J)\xi' = K_0 (\xi', J)\xi' - \langle \xi', \xi' \rangle J \)

If \( J \) is normal to \( \xi' \) and \( \xi' \) has length 1, we obtain:

\[ R(\xi', J)\xi' = + K_0 J \]

and the Jacobi equation becomes

\[ \frac{D^2 J}{dt^2} + K_0 J = 0 \]

Recall that if \( \{ e_1(t), \ldots, e_n(t) \} \) is an free orthornormal frame and if we write \( J = \sum g_i e_i \), then the Jacobi equation becomes:

\[ \forall j, \quad g_{..}^j (t) + \sum_{i=1}^n a_{ij} g_i (t) = 0 \]

where \( a_{ij} = \langle R(\xi', e_i)\xi', e_j \rangle \)

\[ = K_0 \left( -\langle \xi', e_j \rangle \langle \xi', e_i \rangle + \langle \xi', \xi' \rangle \langle e_j, e_i \rangle \right) \]

\[ = K_0 \left( e_{ij} - \langle \xi', e_j \rangle \langle \xi', e_i \rangle \right) \]

Choose \( e_1 = \xi' \), then \( a_{ij} = K_0 (e_{ij} - 5_i \delta_{ij}) \)

and \( g_i = 0 \)
If, for instance \( J = g_2 e_2 \), then

\[
g_2'' + \alpha_2^2 g_2 = 0
\]

and \( \alpha_2 = k_0 \).

So \( g_2'' + \alpha_2^2 g_2 = 0 \)

\( g_2 = \) combination of sine and cosine.

\[ \text{Proposition: (2.4)} \quad \gamma: [0, a] \rightarrow M \]

a geodesic, \( \gamma(0) = q \).

\( J \) a Jacobi field along \( \gamma \) \( \implies \quad J(0) = 0 \).

Denote \( \gamma'(0) = v \), \( \frac{DJ}{dt}(0) = w \).

Then \( J(t) = \exp_q(tw) \tau_v \) \( (tw) \)

\[ \frac{df}{ds}(t, 0) \]

\( f(t, s) = \exp_q \left( \frac{t}{a} \tau_v(s) \right) \)

with \( \tau_v(s) \) acurve in \( T_q \gamma(T_q M) \)

with \( \tau_v(0) = av \) and \( \tau_v'(0) = w \).

Proof: Both vector fields satisfy the Jacobi equation, so they are equal.
If and only if they satisfy the same initial conditions,
\[ J(0) = 0 \quad \text{and} \quad d(\exp q)_{\mathfrak{m}}(0) = 0 \]
\[ \frac{DJ}{dt}(0) = w, \quad \text{we saw earlier that} \quad \frac{d}{dt}(d(\exp q)_{\mathfrak{T}_q M}(tw)) = w. \]

Conjugate points and the critical point of \( \exp \)

A critical point of \( \exp q \) \((q \in M)\) is a point where \( d(\exp q) \) fails to be injective \( \Rightarrow \) \( d(\exp q) \) fails to be injective because \( T_q M \) and \( M \) have the same dim.

\( d(\exp q)_{\mathfrak{m}} \) fails to be injective if \( \exists \, w \in \mathfrak{T}_q (T_q M) \) s.t.
\[ d(\exp q)_{\mathfrak{m}}(w) = 0. \]
Def: Let \( p \in M \) be a point, a point \( q \in M \) is a conjugate point along a geodesic \( \gamma \) if
\[
\gamma : [0, a] \to M.
\]
\[\gamma(0) = p, \exists \ t_0 \text{ s.t. } \gamma(t_0) = q\]
and \( \exists \) Jacobi field along \( \gamma \) s.t.
\[J(0) = 0 = J(t_0)\]

Note: then \( J(t) = \frac{d}{dt} \left( \exp_p \right)_{tv}(t,w)\)
where \( v = \gamma'(0) \) and \( w = \frac{dJ}{dt}(0)\)
and \( J(0) = 0 = J(t_0)\).
So \( t_0 \) is a critical point of \( \lambda(t) = (\exp_p)_{tv}(t,w)\).

Def: The multiplicity of a conjugate point is the number of linearly independent Jacobi fields with the property \( J(0) = J(t_0) = 0 \).

Note that \( \gamma' \) and \( t_0 \) are Jacobi fields, \( \gamma'(0) \neq 0 \) but \( (t\gamma')(0) = 0 \)
\[ \frac{D}{dt} (t \phi') = \phi' + t \frac{D\phi'}{dt} = \phi' \text{ never 0.} \]

There are \( n \) linearly indep. Jacobi fields s.t. \( J(0) = 0 \), one of them is never 0 outside of \( p \), then \( J \) at most \( n-1 \) linearly indep. \( \phi' \) fields that can vanish at \( \phi(t_0) \). So multiplicity \( \leq n-1 \).