More interesting properties of Jacobi fields:

Proposition: \( \gamma : [0, a] \rightarrow M \) geodesic.

If \( J \) Jacobi field along \( \gamma \), then

\[
\left< J(t), \gamma'(t) \right> \text{ is affine linear, i.e.,}
\]

\[
= \left< J(0), \gamma'(0) \right> + \left< \frac{DJ}{dt}(0), \gamma'(0) \right> t
\]

Proof: we need to prove:

\[
\frac{d}{dt} \left< J(t), \gamma'(t) \right> \text{ is constant.}
\]

\[
= \frac{d^2}{dt^2} \left< J(t), \gamma'(t) \right> = \left< \frac{D^2 J}{dt^2}(t), \gamma'(t) \right>
\]

\[= \left< -R(\gamma', J) \gamma', \gamma' \right>(t) = 0 \quad \text{by skew-symmetry of curvature.}
\]

\( \square \).
Corollary 1: \( \langle J(t), \gamma'(t) \rangle \) is either injective or constant. It is constant if \( \langle \frac{DJ}{dt}(0), \gamma'(0) \rangle = 0 \).

Corollary 2: \( \forall t J(t) = 0 \), then \( \langle J(t), \gamma'(t) \rangle = \langle \frac{DJ}{dt}(0), \gamma'(0) \rangle t \)
\[ = \langle w, v \rangle t \]
so \( J(t) \perp \gamma'(t) \ \forall t \Leftrightarrow \langle w, v \rangle = 0 \)

Proposition: \( \gamma: [0, a] \rightarrow M \).
\( \gamma(0) = p \quad \gamma(a) = q \)
Suppose \( q \) is not conjugate to \( p \).

Given \( v_1 \in T_p M \), \( v_2 \in T_q M \)
\( \exists \) Jacobi field \( J \) along \( \gamma \) s.t.
\( J(0) = v_1 \), \( J(a) = v_2 \).

Proof: Given a Jacobi field \( J \) s.t.
\( J(0) = 0 \), define \( ev_{q}(J) := J(a) \).

So we have a map from the space of Jacobi fields, \( J_{op} \), to \( T_q M \).
evq: $T_{q\theta} \rightarrow T_q M$

this is linear between vector spaces of dim. n.

Claim: evq is an isom.
so it is injective.

$evq(J) = 0$, means $T(a) = 0$
but $q$ is not conjugate to $p$, so $J \equiv 0$.

So evq is injective, hence also surjective, hence $\exists J_2$

s.t. $J_2(0) = 0$, $J_2(a) = V_2$.

Now reverse $\theta$ to obtain $J_1$, s.t.

$J_1(0) = V_1$, $J_1(a) = 0$.

Then define $J = J_1 + J_2$

Remark: $J$ is unique because if

we had $J, J'$, s.t. then $(J - J')(0) = (E - J')(a)$

but $q$ is NOT conjugate to $p$. $= 0$
**Corollary:** \( \gamma : [0, a) \rightarrow M.\)  
\( J^+ := \) space of Jacobi fields, \( J \)
\( n.t. \ J(0) = 0 \) \& \( \frac{dJ}{dt}(0) \perp \gamma'(0). \)

Let \( \{J_1, \ldots, J_{n-1}\} \) be a basis of \( J^+ \), then if \( \gamma(t) \) is not conjugate to \( \gamma(0) \), then for \( \forall t \):
\( \{J_1(t), \ldots, J_{n-1}(t)\} \) is a basis of \( \gamma'(t)^\perp \)

**Proof:** From the first prop.:
\[ \langle J(t), \gamma'(t) \rangle = \langle \frac{dJ}{dt}(0), \gamma'(0) \rangle t \]
So \( J(t) \perp \gamma'(t) \iff \frac{dJ}{dt}(0) \perp \gamma'(0) \) if \( t \neq 0. \)

Also, by the proof of the second prop.:
\( e_{N_t} : J_{0,p} \rightarrow T_{\gamma(t)}M \)
is an isom. if \( \gamma(t) \) is not conjugate to \( \gamma(0) = p \)
So $J_1(t), \ldots, J_{n-1}(t) \in \mathcal{X}(t)^1$ and they are linearly indep. if $r(t)$ is not conjugate to $\mathcal{X}(o) = \mathfrak{p}$. □