Prop. More <u>interesting properties of Jacobi fields</u>:

<u>Proposition:</u> $\gamma : [0,a] \longrightarrow M$ geodesic. J Jacobi field along γ.

then

$\langle J(t), \gamma'(t) \rangle$ is affine linear in t, i.e.,

$$= \langle J(0), \gamma'(0) \rangle + \left\langle \frac{DJ}{dt}(0), \gamma'(0) \right\rangle t$$

<u>Proof:</u> need to prove:

$$\frac{d}{dt} \langle J(t), \gamma'(t) \rangle \text{ is constant.}$$

or

$$\frac{d^2}{dt^2} \langle J(t), \gamma'(t) \rangle \equiv 0.$$

$$\frac{d^2}{dt^2} \langle J(t), \gamma'(t) \rangle = \left\langle \frac{D^2 J}{dt^2}(t), \gamma'(t) \right\rangle$$

$$= \langle -R(\gamma', J)\gamma', \gamma' \rangle(t) = 0 \quad \forall t$$

by skew-symmetry of curvature.

$\square$.

Corollary 1: $\langle J(t), \gamma'(t)\rangle$ is either injective or constant. It is constant $\iff \langle \frac{DJ}{dt}(0), \gamma'(0)\rangle = 0$

Corollary 2: If $J(0)=0$, then

$$\langle J(t), \gamma'(t)\rangle = \langle \tfrac{DJ}{dt}(0), \gamma'(0)\rangle t$$
$$= \langle w, v\rangle t$$

So $J(t) \perp \gamma'(t)\ \forall t \iff \langle w, v\rangle = 0$

Proposition: $\gamma : [0,a] \longrightarrow M$.
$\gamma(0) = p \qquad \gamma(a) = q$

Suppose q is not conjugate to p. Given $V_1 \in T_pM$, $V_2 \in T_qM$ $\exists$ Jacobi field J along γ s.t.

$$J(0) = V_1, \quad J(a) = V_2.$$

Proof: Given a Jacobi field J s.t. $J(0)=0$, define $ev_q(J) := J(a)$

so we have a map from the space of Jacobi fields $\mathcal{J}_{0,p}$ to T_qM

$ev_q : \mathcal{J}_{qp} \longrightarrow T_q M$

this is linear between vector spaces of dim. n.

<u>Claim</u>: ev_q is an isom.

or it is injective.

If $ev_q(J) = 0$, means $J(a) = 0$

but q is <u>not</u> conjugate to p, so

$J \equiv 0$.

So ev_q is injective, hence also surjective, hence $\exists\, J_2$ s.t. $J_2(0) = 0$, $J_2(a) = V_2$.

Now reverse γ to obtain J_1 s.t.

$$J_1(0) = V_1 \quad , \quad J_1(a) = 0 .$$

Then define $J := J_1 + J_2$

$\square$

<u>Remark</u>: J is unique because if we had J, J', then $(J - J')(0) = (J - J')(a) = 0$

but q is NOT conjugate to p.

<u>Corollary</u>: $\gamma : [0,a] \longrightarrow M$.

$\mathcal{J}^{\perp} :=$ space of Jacobi fields J s.t. $J(0)=0$ & $\frac{DJ}{dt}(0) \perp \gamma'(0)$

Let $\{J_1, \dots, J_{n-1}\}$ be a basis of $\mathcal{J}^{\perp}$, then $\forall t$ if $\gamma(t)$ is not conjug. to $\gamma(0)$, ~~then $\forall t$~~ then

$\{J_1(t), \dots, J_{n-1}(t)\}$ is a basis of $\gamma'(t)^{\perp}$

<u>Proof</u>: From the first prop.:

$$\langle J(t), \gamma'(t) \rangle = \langle \tfrac{DJ}{dt}(0), \gamma'(0) \rangle t$$

So $J(t) \perp \gamma'(t) \iff \frac{DJ}{dt}(0) \perp \gamma'(0)$ if $t \neq 0$.

Also by the proof of the second prop.:

$$ev_t : \mathcal{J}_{0,p} \longrightarrow T_{\gamma(t)}M$$

is an isom. if $\gamma(t)$ is not conjugate to $\gamma(0) = p$

So $J_1(t), \ldots, J_{n-1}(t) \in \gamma'(t)^{\perp}$ and they are linearly indep. if $\gamma(t)$ is not conjugate to $\gamma(0) = p$.

$\square$