A few remarks: \[ M \to \overline{M} \]

\[ \dim n \quad \dim m + n \]

\[ T\overline{M} := \bigsqcup_{x \in \overline{M}} T_x \overline{M} \]

\[ T\overline{M} |_{\mathcal{M}} := \bigsqcup_{x \in \mathcal{M}} T_x \overline{M} \]

More generally, one can restrict any vector bundle on \( \overline{M} \) to \( M \):

\[ E \mid_{\overline{M}} \to E|_{\mathcal{M}} \]

\( E|_{\mathcal{M}} \) will be a vector bundle on \( M \).

Recall: \( \exists \) covering \( \overline{M} = \bigcup_{i \in I} U_i \)

s.t. on each \( U_i \): \( E \mid_{U_i} \cong U_i \times \mathbb{R}^n \)

where \( n \) is the rank of \( E \).

and we have the gluing data: on \( U_i \cap U_j \)
$g_{ij}(x) \in \mathfrak{gl}_n(\mathbb{R})$, $x \in U_{i,j}$

$U_i \times \mathbb{R}^n \ni (x, v) \mapsto (x, g_{ij}(x)v) \in U_j \times \mathbb{R}^n$

all the entries of $g_{ij}$ one differentiable functions of $x$.

recall that we also have a cocycle condition on triple intersections:

$U_i \cap U_j \cap U_k : g_{ij} \circ g_{jk} \circ g_{ki} = \text{Id}.$

The restriction $E|_M$ will be given by

$M = \bigcup_{i \in I} M \cap U_i$

If the gluing data will be:

$g_{ij} |_{U_i \cap U_j \cap M}$

In our situation, $TM|_M$ is a vector bundle on $M$. 
We have \( TM \subset T\bar{M} \). 

We can form:

\[
M_{M/\bar{M}} := \frac{T\bar{M}}{M} \bigg/ \frac{T\bar{M}}{TM}
\]

\[
= \bigsqcup_{x \in \bar{M}} \frac{T_x\bar{M}}{T_xM}
\]

To see this is a vector bundle, we need trivializations (i.e., \( \cong U \times \mathbb{R}^n \)) and gluing data.

To obtain these, use the result from last quarter saying that \( \exists \) coordinate charts \( U \) of \( \bar{M} \) at every point:

\[
\varphi_U : U \rightarrow \bar{M}
\]

\[
(\bar{x}_1, \ldots, \bar{x}_{m+n}) \mapsto \varphi_U(\bar{x}_1, \ldots, \bar{x}_{m+n})
\]

s.t. \( M \cap \varphi(U) = \text{image of } \text{zeros of } \).
In the case where we have a metric on $\overline{M}$, we can decompose:

$$T_x \overline{M} = T_x M \oplus (T_x M)^\perp$$

$\forall x \in M$

So we have an isomorphism:

$$\overline{\left(T_x M\right)^\perp} \xrightarrow{\cong} (N_{M/\overline{M}})_x.$$ 

and

$$N_{M/\overline{M}} \cong (TM)^\perp.$$ 

We saw:

$$S_\eta(x) = -\left(\overline{\nabla_x N}\right)^T.$$ 

Now we consider the normal comp.

$$\left(\overline{\nabla_x N}\right)^N = \overline{\nabla_x^\perp N}$$

where $\eta$ is a normal vector to $M$ at $p \in M$, $x \in T_p M$, $N$ extends $\eta$ to a vector field on $\overline{M}$, normal to $M$.
Note that: \[ \tilde{\nabla}_x^+ N = \tilde{\nabla}_x^+ N - (\tilde{\nabla}_x^+ N)^T \]
\[ = \tilde{\nabla}_x N + S_q (x) \]

We can verify that \( \tilde{\nabla}^\perp \) has all the properties of a connection except that the input is a vector field and a normal field and the output is a normal field. We can define its curvature, which is called the normal curvature:

\[ R^+(x,y)Z := -\nabla_x^\perp \nabla_y^\perp Z + \nabla_y^\perp \nabla_x^\perp Z \]
\[ + \nabla_{[x,y]}^\perp Z \]

**Theorem:** (The fundamental equations)

1. \( \langle \tilde{R}(x,y)Z,T \rangle \quad \langle R(x,y)Z,T \rangle \quad = \langle R(x,y)Z,T \rangle - \langle B(y,T),B(x,Z) \rangle \)
(2) **Ricci equations:**

\[
\langle \bar{R}(x,y)N, N' \rangle = \langle R^+(x,y)N, N' \rangle = \langle [S_w, S_w]X, Y \rangle
\]