<u>A few remarks:</u> $M \hookrightarrow \bar{M}$

dim n dim m+n

$$T\bar{M} := \coprod_{x \in \bar{M}} T_x \bar{M}$$

$$T\bar{M}\big|_M := \coprod_{x \in M} T_x \bar{M}$$

More generally, one can restrict any vector bundle on $\bar{M}$ to M:

$$E \text{ on } \bar{M} \rightsquigarrow E\big|_M$$

$E\big|_M$ will be a vector bundle on M.

<u>Recall</u>: $\exists$ covering $\bar{M} = \bigcup_{i \in I} U_i$

s.t. on each U_i $\quad E\big|_{U_i} \cong U_i \times \mathbb{R}^r$

where r is the rank of E.

and we have the gluing data:

on $U_i \cap U_j$

$$g_{ij}(x) \in GL_n(\mathbb{R}). \quad x \in U_i \cap U_j$$

$$U_i \times \mathbb{R}^n \ni (x, v) \longmapsto (x, g_{ij}(x)v) \in U_j \times \mathbb{R}^n$$

all the entries of g_{ij} are differentiable functions of x.

recall that we also have a cocycle condition on triple intersections: $U_i \cap U_j \cap U_k$:

$$g_{ij} \circ g_{jk} \circ g_{ki} = Id.$$

The restriction $E|_M$ will be given by $M = \bigcup_{i \in I} M \cap U_i$

the gluing data will be:

$$g_{ij}|_{U_i \cap U_j \cap M}$$

In our situation, $T\bar{M}|_M$ is a vector bundle on M.

We have $TM \subset T\bar{M}|_M$

We can form:

$$N_{M/\bar{M}} := T\bar{M}|_M \big/ TM$$

$$:= \coprod_{x \in M} \left(T_x\bar{M} \big/ T_xM \right)$$

To see this is a vector bundle, we need trivializations (i.e., $\cong U_i \times \mathbb{R}^n$) and gluing data.

To obtain these, use the result from last quarter saying that $\exists$ coordinate charts U of $\bar{M}$ at every point:

$$\varphi_U : U \longrightarrow \bar{M}$$

$$(x_1, \dots, x_{m+n}) \longmapsto \varphi_U(x_1, \dots, x_{m+n})$$

s.t. $M \cap \varphi(U) =$ image of zeros of

$x_{m+1}, \dots, x_{m+n}$

In the case where we have a metric on $\bar{M}$, we can decompose:

$$T_x \bar{M} = T_x M \oplus (T_x M)^{\perp}$$

$$\forall\ x \in M$$

So we have an isom:

$$(T_x M)^{\perp} \xrightarrow{\cong} (N_{M/\bar{M}})_x .$$

and $$N_{M/\bar{M}} \cong (TM)^{\perp}$$

<u>We saw</u>: $$S_\eta(x) = -(\bar{\nabla}_x N)^T$$

Now we consider the normal comp.

$$(\bar{\nabla}_x N)^N =: \bar{\nabla}_x^{\perp} N$$

where η is a normal vector to M at $p \in M$, $x \in T_p M$, N extends η to a vector field on $\bar{M}$, normal to M

<u>Note that</u>: $\bar{\nabla}_x^{\perp} N = \bar{\nabla}_x N - (\bar{\nabla}_x N)^T$

$$= \bar{\nabla}_x N + S_\eta(x)$$

We can verify that $\nabla^{\perp}$ has all the properties of a connection except that the input is a vector field and a normal field and the output is a normal field. We can define its curvature, which is called the normal curvature:

$$R^{\perp}(X,Y)Z := -\nabla_X^{\perp}\nabla_Y^{\perp} Z + \nabla_Y^{\perp}\nabla_X^{\perp} Z + \nabla_{[X,Y]}^{\perp} Z$$

<u>Theorem</u>: (The fundamental equations)

(1) <u>Gauss</u>: $\langle \bar{R}(X,Y)Z, T\rangle$

$= \langle R(X,Y)Z, T\rangle - \langle B(Y,T), B(X,Z)\rangle$

$$+\langle B(X,T), B(Y,Z)\rangle$$

(2) <u>Ricci equations:</u>

$$\langle \bar{R}(X,Y)N, N'\rangle - \langle R^{\perp}(X,Y)N, N'\rangle =$$

$$= \langle [S_N, S_{N'}]X, Y\rangle$$