

$$+ \langle B(X, T), B(Y, Z) \rangle$$

(2) Ricci equations:

$$\begin{aligned} \langle \bar{R}(X, Y)N, N' \rangle &= \langle R^+(X, Y)N, N' \rangle = \\ &= \langle [S_N, S_{N'}]X, Y \rangle \end{aligned}$$

Theorem: (Codazzi equations)

$$\begin{aligned} \langle \bar{R}(X, Y)Z, \eta \rangle &= (\bar{\nabla}_Y B)(X, Z, \eta) \\ &\quad - (\bar{\nabla}_X B)(Y, Z, \eta) \end{aligned}$$

* $X, Y, Z \in \mathcal{X}(M)$, η normal vector field to M

Notation: Here:

$$\text{def } B(X, Y, \eta) = \langle B(X, Y), \eta \rangle$$

$$B(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$$

$$\begin{aligned} (\bar{\nabla}_X B)(Y, Z, \eta) &= X(B(Y, Z, \eta)) - B(\bar{\nabla}_X Y, Z, \eta) \\ &\quad - B(Y, \bar{\nabla}_X Z, \eta) - B(Y, Z, \nabla_X^\perp \eta) \end{aligned}$$

Remark: The second fundamental form is a quadratic form on TM with values in $N_{M/\bar{M}}$:

$$B : TM \otimes TM \longrightarrow N_{M/\bar{M}}$$

$$X \otimes Y \longmapsto B(X, Y)$$

$$\text{or } S^2 TM \longrightarrow N_{M/\bar{M}}.$$

If $N_{M/\bar{M}}$ has rank 1, then at each $x \in M$, we have a symmetric bilinear form on $T_x M$.

Proof of the Gauss equations:

$$\bar{R}(X, Y)Z = \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_X \bar{\nabla}_Y Z + \bar{\nabla}_{[X, Y]} Z$$

$$\text{we know } \bar{\nabla}_X Z = \nabla_X Z + B(X, Z)$$

So

$$\begin{aligned} \bar{R}(X, Y)Z &= \bar{\nabla}_Y (\nabla_X Z + B(X, Z)) \\ &\quad - \bar{\nabla}_X (\nabla_Y Z + B(Y, Z)) \\ &\quad + \nabla_{[X, Y]} Z + B([X, Y], Z) \end{aligned}$$

$$\begin{aligned}
&= \nabla_Y (\nabla_X Z + B(X, Z)) + B(Y, \nabla_X Z + B(X, Z)) \\
&- \nabla_X (\nabla_Y Z + B(Y, Z)) - B(X, \nabla_Y Z + B(Y, Z)) \\
&\quad + \nabla_{[X, Y]} Z + B([X, Y], Z).
\end{aligned}$$

$$\begin{aligned}
&= R(X, Y)Z + \nabla_Y B(X, Z) + B(Y, \nabla_X Z) \\
&\quad + B(Y, B(X, Z)) - \nabla_X B(Y, Z) - B(X, \nabla_Y Z) \\
&\quad - B(X, B(Y, Z)) + B([X, Y], Z)
\end{aligned}$$

Note: $B(X, Z)$ & $B(Y, Z)$ are normal vector fields so :

$$\begin{aligned}
\nabla_Y B(X, Z) &= \nabla_Y^\perp B(X, Z) - S_{B(X, Z)}^{(Y)} \\
&\parallel \\
\nabla_Y B(X, Z) + B(Y, B(X, Z))
\end{aligned}$$

So the expression above is equal to:

$$\begin{aligned}
&= R(X, Y)Z + \nabla_Y^\perp B(X, Z) - S_{B(X, Z)}^{(Y)} \\
&\quad - \nabla_X^\perp B(Y, Z) + S_{B(Y, Z)}^{(X)} \\
&\quad + B(Y, \nabla_X Z) - B(X, \nabla_Y Z) + B([X, Y], Z)
\end{aligned}$$

Now pair with T :

$$\begin{aligned}
 & \langle \bar{R}(X, Y)Z, T \rangle = \langle R(X, Y)Z, T \rangle \\
 & + \langle \nabla_Y^\perp B(X, Z), T \rangle - \langle S_{B(X, Z)}(Y), T \rangle \\
 & - \langle \nabla_X^\perp B(Y, Z), T \rangle + \langle S_{B(Y, Z)}(X), T \rangle \\
 & + \cancel{\langle B(Y, \nabla_X Z), T \rangle} - \cancel{\langle B(X, \nabla_Y Z), T \rangle} \\
 & + \langle B([X, Y], Z), T \rangle \\
 & = 0 \quad \text{because } \nabla_X^\perp \text{ takes values in} \\
 & N_{M/\bar{M}} \text{ and } T \text{ is tangent} \\
 & \text{to } M
 \end{aligned}$$

recall: $\langle B(X, Y), \eta \rangle = \langle S_\eta(X), Y \rangle$

$$\begin{aligned}
 & = \langle S_Y(X), X \rangle
 \end{aligned}$$

so $\langle S_{B(X, Z)}(Y), T \rangle = \langle B(Y, T), B(X, Z) \rangle$

and $\langle S_{B(Y, Z)}(X), T \rangle = \langle B(X, T), B(Y, Z) \rangle$

Proof of the Ricci equations:

$$\bar{R}(X, Y)N = \bar{\nabla}_Y \bar{\nabla}_X N - \bar{\nabla}_X \bar{\nabla}_Y N + \bar{\nabla}_{[X, Y]} N$$

$$= \bar{\nabla}_y (\nabla_x^+ N - S_N(x)) - \bar{\nabla}_x (\nabla_y^+ N - S_N(y)) \\ + \nabla_{[x,y]}^\perp N - S_N([x,y])$$

$$= \nabla_y^\perp (\nabla_x^\perp N - S_N(x)) - S_{\nabla_x^\perp N - S_N(x)}(y) \\ - \nabla_x^+ (\nabla_y^\perp N - S_N(y)) + S_{\nabla_y^\perp N - S_N(y)}(x) \\ + \nabla_{[x,y]}^\perp N - S_N([x,y])$$

$$= R^+(x,y)N - \nabla_y^\perp S_N(x) - S_{\nabla_x^+ N}(y) \\ + S_{S_N(x)}(y) + \nabla_x^+ S_N(y) + S_{\nabla_y^+ N}(x) \\ - S_{S_N(y)}(x) - S_N([x,y])$$

$$= R^+(x,y)N - \bar{\nabla}_y S_N(x) - S_{S_N(x)}(y) \\ - S_{\nabla_x^\perp N}(y) + S_{S_N(x)}(y) + \bar{\nabla}_x S_N(y) \\ + S_{\nabla_y^+ N}(x) + S_{S_N(y)}(x) - S_{S_N(y)}(x) \\ - S_N([x,y])$$

$$= R^\perp(x, y)N - \bar{\nabla}_y S_N(x) + \bar{\nabla}_x S_N(y) \\ - S_{\nabla_x^\perp N}(y) + S_{\nabla_y^\perp N}(x) - S_N([x, y])$$

Now pair with N :

$$= R^\perp(x, y)N - \nabla_y S_N(x) - B(y, S_N(x)) \\ + \nabla_x S_N(y) + B(x, S_N(y)) - S_{\nabla_x^\perp N}(y) \\ + S_{\nabla_x^\perp N}(x) - S_N([x, y])$$

Now pair with N' :

$$\langle R(x, y)N, N' \rangle = \langle R^\perp(x, y)N, N' \rangle \\ - \langle \nabla_y S_N(x), N' \rangle \stackrel{=0}{=} + \langle \nabla_x S_N(y), N' \rangle \stackrel{=0}{=} \\ - \langle B(y, S_N(x)), N' \rangle + \langle B(x, S_N(y)), N' \rangle \\ - \langle S_{\nabla_x^\perp N}(y), N' \rangle + \langle S_{\nabla_x^\perp N}(x), N' \rangle \\ - \langle S_N([x, y]), N' \rangle \stackrel{=0}{=}$$

S is tangential

$$= \langle R^\perp(x, y)N, N' \rangle \\ - \langle S_N(x), S_{N'}(y) \rangle + \langle S_N(y), S_{N'}(x) \rangle$$

$$\begin{aligned}
 &= \langle R^\perp(x, y) N, N' \rangle \\
 &= \langle S_N, S_N(x), y \rangle + \langle S_N \circ S_{N'}(x), y \rangle \\
 &= \langle R^\perp(x, y) N, N' \rangle \\
 &\quad + \langle [S_N, S_{N'}](x), y \rangle \quad \square
 \end{aligned}$$