Manifolds with constant curvature

$M$ has constant curvature means that the sectional curvature is independent of the point and the two dimensional space in which it is computed.

One can compute that if the metric is multiplied by a constant $c$, then the sectional curvature is multiplied by $\frac{1}{c}$.

So we can assume that the constant sectional curvature is $K = 0$ or $\pm 1$.

Theorem: If $M$ is complete, simply connected with constant curvature $= 0, \pm 1$, then
\[ M \cong \mathbb{R}^n \quad \text{if} \quad K = 0 \]
\[ M \cong S^n \quad \text{if} \quad K = 1 \]
\[ M \cong H^n \quad \text{if} \quad K = -1 \]

\underline{Preliminaries:}

Suppose \( \tilde{M} \) and \( M \) are Riemannian manifolds of dim. \( n \). Choose an isometry \( i : T_\mathbf{p} M \to T_\mathbf{f} \tilde{M} \) where \( \mathbf{p} \in M \), \( \mathbf{f} \in \tilde{M} \).

Choose \( V \subseteq M \) s.t. \( \mathbf{f} \in V \) and \( V \) is a normal neighborhood of \( \mathbf{f} \) s.t. \( \exp_\mathbf{f} (V) \subset T_\mathbf{f} \tilde{M} \).

\[ \exp_\mathbf{p} \downarrow \]
\[ \mathbf{p} \in V \subseteq M \quad \tilde{M} \quad \tilde{W} \]

Assume \( \exp_\mathbf{p} \) is well-defined on \[ i \left( \exp_\mathbf{p} (V) \right) = \tilde{W} \]
\[ f := \exp_p \circ \iota \circ \exp_p^{-1} : V \rightarrow W \]

is a diffeomorphism.

Next, for \( q \in V \), choose a unit speed geodesic \( \gamma : [0, l] \rightarrow V \)

\[ \gamma(0) = \iota, \quad \gamma(l) = q \]

(where \( l = d(\iota, q) \))

then \( f \circ \gamma : \gamma : [0, l] \rightarrow W \) is the unique geodesic from \( \gamma \) to unit speed \( \tilde{q} = f(q) \)

Define \( \eta_t : T_{\gamma(t)} M \rightarrow T_{\gamma(t)} \tilde{M} \)

as \( \eta_t(v) = \) parallel transport along \( \gamma \) back to \( \iota \) of \( v \), then take the image by \( \iota \), then parallel transport to \( \gamma(t) \).

(Note: \( \iota \) \( \eta_t \) is an isometry.)