Manifolds with constant curvature

M has constant curvature means ~~that~~ the sectional curvature is independent of the point and the two dimensional space on which it is computed.

One can compute that if the metric is multiplied by a constant $c>0$, then the sectional curvature is multiplied by $\frac{1}{c}$.

So we can assume that the constant sectional curvature is

$$K = 0 \text{ or } \pm 1$$

<u>Theorem</u>: If M is complete, <u>simply connected</u> with constant curvature $= 0, \pm 1$, then

$M \approx \mathbb{R}^n$ if $K=0$

$M \approx S^n$ if $K=1$

$M \approx \mathbb{H}^n$ if $K=-1$

Preliminaries:

Suppose $\tilde{M}$ and M are Riemannian manifolds of dim. n. Choose an isometry $i: T_pM \to T_{\tilde{p}}\tilde{M}$ where $p \in M$, $\tilde{p} \in \tilde{M}$.

Choose $V \subset M$ s.t. $p \in V$ and V is a normal neighborhood of p.

$$\begin{tikzcd}
0 \in \exp_p^{-1}(V) \arrow[r, subset] \arrow[d, mapsto, "\exp_p"'] & T_pM \arrow[r, "i"] & T_{\tilde{p}}\tilde{M} \supset \tilde{W} \arrow[d] \\
p \in V \arrow[r, subset] & M & \tilde{M} \supset W
\end{tikzcd}$$

assume $\exp_{\tilde{p}}$ is well-defined on $i(\exp_p^{-1}(V)) =: \tilde{W}$

$f := \exp_{\tilde{p}} \circ i \circ \exp_p^{-1} : V \longrightarrow W.$

is a diffeomorphism.

Next for $q \in V$, choose a unit speed geodesic $\gamma : [0, \ell] \to V$ s.t. $\gamma(0) = p$, $\gamma(\ell) = q$

(where $\ell = d(p, q)$)

then $f \circ \gamma := \tilde{\gamma} : [0, \ell] \longrightarrow W$ is the unique unit speed geodesic from $\tilde{p}$ to $\tilde{q} := f(q)$

Define $\varphi_t : T_{\gamma(t)} M \longrightarrow T_{\tilde{\gamma}(t)} \tilde{M}$

as $\varphi_t(v) =$ parallel transport along γ back to p of v, then take the image by i, then parallel transport to $\tilde{\gamma}(t)$.

Note: $\forall t$ φ_t is an isometry.