

In the notation of the book:

$P_t :=$ parallel transport along γ
from γ to $\gamma(t)$.

$\tilde{P}_t :=$ parallel transport along $\tilde{\gamma}$
from $\tilde{\gamma}$ to $\tilde{\gamma}(t)$.

$P_t : T_{\gamma} M \rightarrow T_{\gamma(t)} M \quad \} \text{isometric}$

$\tilde{P}_t : T_{\tilde{\gamma}} \tilde{M} \rightarrow T_{\tilde{\gamma}(t)} \tilde{M}$

then $\varphi_t = \tilde{P}_t \circ i \circ P_t^{-1} : T_{\gamma(t)} M \rightarrow T_{\tilde{\gamma}(t)} \tilde{M}$

Theorem (2.1 Chapter 8): If $\forall x, y, u, v$
 ~~$\in T_{\gamma} M$~~ , we have $\in T_{\gamma(t)} M$,

$$\cancel{\langle R(x, y)u, v \rangle} = \cancel{\langle \tilde{R}(i(x), i(y))i(u), i(v) \rangle}$$

$$\langle R(x, y)u, v \rangle = \langle \tilde{R}(\varphi_t(x), \varphi_t(y))\varphi_t(u), \varphi_t(v) \rangle$$

then $f : V \rightarrow W$ is an ~~local~~ isometry

and $df_{\gamma} = i$.

(In fact $df_{\gamma(t)} = \varphi_t$).
 $\forall t.$

Proof: We want to prove:

$$\frac{df}{f\gamma(t)} = \varphi_t.$$

We have $f = \exp_p^{-1} \circ i \circ \exp_p$

so $\frac{df}{f\gamma(t)} = d(\exp_p) \circ i \circ d(\exp_p^{-1})$

Recall that any Jacobi field on γ
s.t. $J(0) = 0$ is of the form:

$$J(t) = d(\exp_{t\gamma'(0)}) \left(t \frac{DJ}{dt}(0) \right)$$

Let $q = \gamma(l)$, ~~choose~~ $v \in T_q M$.

$\exists!$ Jacobi field J on γ s.t.

$$J(0) = 0 \quad J(l) = v.$$

define \tilde{J} via $\tilde{J}(t) := \varphi_t(J(t))$

Claim: the equality of curvatures
implies that \tilde{J} is a Jacobi field
along $\tilde{\gamma}$.

We show that \tilde{J} satisfies the

Jacobi equation:

$$\frac{D^2 \tilde{J}}{dt^2} + \tilde{R}(\tilde{\gamma}'(t), \tilde{J}(t)) \tilde{\gamma}'(t) = 0$$

we know: $\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t)) \gamma'(t) = 0$

Compute in an orthonormal basis:

Let e_1, \dots, e_n be an orthonormal basis of $T_p M$, parallel transport them to obtain orthonormal bases $e_1(t), \dots, e_n(t)$ of $T_{\gamma(t)} M$, $\forall t$.

Define $\tilde{e}_i \cdot (t) = \psi_t(e_i \cdot (t))$.

Write $J(t) = \sum_{i=1}^n y_i \cdot e_i(t)$

we have $\tilde{J}(t) = \sum_{i=1}^n y_i \cdot \tilde{e}_i(t)$

(because ψ_t is linear w.r.t t .)

$$\frac{D^2 J}{dt^2} = \sum_{i=1}^n \ddot{y}_i \cdot e_i(t)$$

and $\frac{D^2 \tilde{J}}{dt^2} = \sum_{i=1}^n \ddot{y}_i \cdot \tilde{e}_i(t)$

The Jacobi equation is equivalent to

$$\forall i : 0 = \gamma''_i + \langle R(\gamma'(t), J(t))\gamma'(t), e_i(t) \rangle \\ = \gamma''_i + \langle \tilde{R}(\tilde{\gamma}'(t), \tilde{J}(t))\tilde{\gamma}'(t), \tilde{e}_i(t) \rangle$$

So \tilde{J} is a Jacobi field.

$$\text{So } \tilde{J}(t) = d(\exp_{t\tilde{\gamma}'(0)})\left(t \frac{D\tilde{J}}{dt}(0)\right) \\ = \Phi\varphi_t(J(t))$$

$$= \varphi_t\left(d(\exp_{t\gamma'(0)})\left(t \frac{DJ}{dt}(0)\right)\right)$$

④ recall $\varphi_0 = i$, so.

$$\frac{D\tilde{J}}{dt}(0) = \varphi_0\left(\frac{DJ}{dt}(0)\right)$$

$$\text{also } \tilde{\gamma}'(0) = i\gamma'(0) = i\left(\frac{DJ}{dt}(0)\right)$$

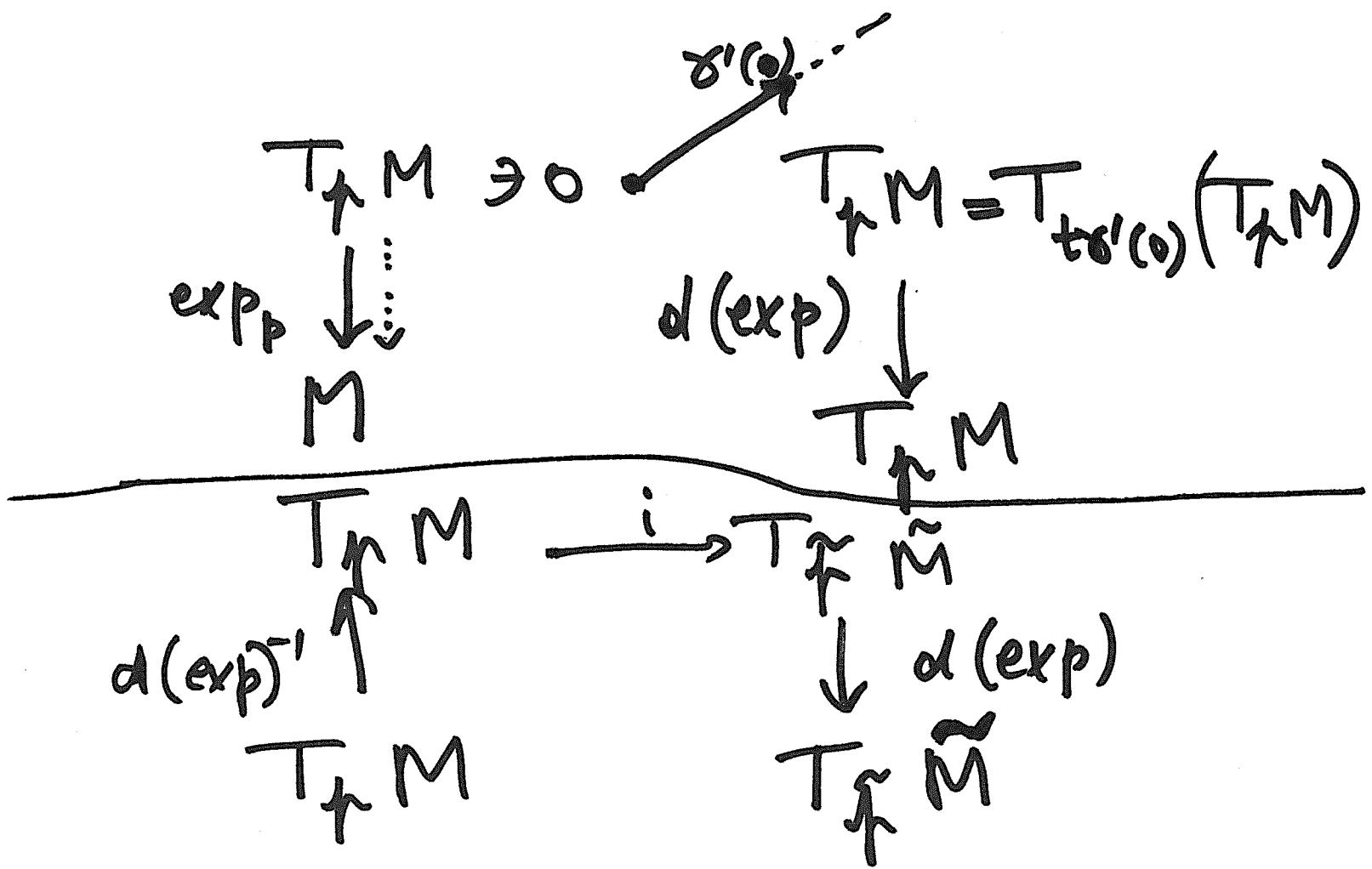
Therefore: $d(\exp_{t\tilde{\gamma}'(0)})$

$$d(\exp_{t\tilde{\gamma}'(0)})\left(t i \frac{DJ}{dt}(0)\right) = \\ = \varphi_t\left(d(\exp_{t\gamma'(0)})\right)\left(t \frac{DJ}{dt}(0)\right)$$

$$5_0 \quad \psi_t = d(\exp_{t\delta'(0)}) \circ d(\exp_{t\delta'(0)}^{-1})$$

(take $\frac{D}{dt}(0) = d(\exp_{t\delta'(0)})^{-1}(\text{something})$)

Now plug in $t=0$ to get $d\psi_p = i \square$



Corollaries: Given two manifolds M, \tilde{M} of the same dim. and equal constant curvature, $p \in M, \tilde{p} \in \tilde{M}$

and any isometry $i: T_p M \rightarrow T_{\tilde{p}} \tilde{M}$,
 \exists neighborhood V of p
& a neighborhood W of \tilde{p}
and an isometry $f: V \xrightarrow{\cong} W$
s.t. $d_{T_p} f = i$

② Same result, but take $M = \tilde{M}$,
 $p = \tilde{p} \therefore$ this means that
manifolds with constant curvature
have a lot of local isometries.

Hyperbolic n-space: H^n

as a differentiable manifold.

$$H^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}.$$

We endow this with the metric

$$g = \sum_{i=1}^n \frac{dx_i^2}{x_n^2}$$

at any point $\alpha = (a_1, \dots, a_n) \in H^n$, we can

identify: $T_a H^n$ with R^n .

then $g_{ij} = \frac{\delta_{ij}}{a_n^2}$

$$(d\varphi_{x_i})(\underbrace{t_1, \dots, t_n}_n) = t_i$$

Claim: H^n is a complete Riemannian manifold.

We prove that geodesics are defined on all of R .

Claim: The geodesics are the vertical half lines and the vertical half circles with center on $\{x_n=0\}$.

① First we prove that these are geodesics: