In the notation of the book:

\[ P_t := \text{parallel transport along } \gamma \text{ from } \gamma(0) \text{ to } \gamma(t) \]

\[ \tilde{P}_t := \text{parallel transport along } \tilde{\gamma} \text{ from } \tilde{\gamma}(0) \text{ to } \tilde{\gamma}(t) \]

\[ P_t : T_{\gamma(0)}M \rightarrow T_{\gamma(t)}M \] isometric

\[ \tilde{P}_t : T_{\tilde{\gamma}(0)}\tilde{M} \rightarrow T_{\tilde{\gamma}(t)}\tilde{M} \]

then \[ \Psi_t = \tilde{P}_t \circ \circ P_t : T_{\gamma(t)}M \rightarrow T_{\tilde{\gamma}(t)}\tilde{M} \]

**Theorem (2.1 Chapter 8):** If \( x, y, u, v \in T_{\gamma(0)}M \), we have \( \langle x, y \rangle = \langle \tilde{R}(x, y), u, v \rangle , \)

\[ \langle R(x, y)u, v \rangle = \langle \tilde{R}(x, y), u, v \rangle \]

\[ \langle R(x, y)u, v \rangle = \langle \tilde{R}(\gamma_t(x), \gamma_t(y)), \gamma_t(u), \gamma_t(v) \rangle \]

then \( f : V \rightarrow W \) is an isometric and \( df_t = i . \)

(The fact \( df_t(\gamma_t) = \gamma_t \).)
Proof: We want to prove:
\[ df_t(x(t)) = y_t. \]

We have \( f = \exp_p \circ \exp_p^{-1} \),
so
\[ df_{x(t)} = d(\exp_p) \circ i \circ d(\exp_p^{-1}). \]

Recall that any Jacobi field \( J \) on \( \gamma \) s.t. \( J(0) = 0 \) is of the form:
\[ J(t) = d(\exp_{t \gamma'(0)}) (t \frac{dJ}{dt}(0)). \]

Let \( q = \varphi(l) \), choose \( v \in T_{qM} \).

∃! Jacobi field \( J \) on \( \gamma \) s.t.
\[ J(0) = 0 \quad J(l) = v. \]

Define \( \tilde{\gamma} \) via \( \tilde{\gamma}(t) := \gamma_t(J(t)) \)

Claim: the equality of curvatures implies that \( \tilde{\gamma} \) is a Jacobi field along \( \gamma \).

We show that \( \tilde{\gamma} \) satisfies the
Jauli equation:

\[
\frac{D^2 \tilde{\mathbf{r}}}{dt^2} + R(\tilde{\mathbf{r}}(t), \tilde{\mathbf{J}}(t)) \tilde{\mathbf{r}}'(t) = 0
\]

we know:

\[
\frac{D^2 \tilde{\mathbf{r}}}{dt^2} + R(\tilde{\mathbf{r}}'(t), \tilde{\mathbf{J}}(t)) \tilde{\mathbf{r}}'(t) = 0
\]

Compute in an orthonormal basis:

Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( T_p M \), parallel transport them to obtain orthonormal bases \( e_i(t), \ldots, e_n(t) \) of \( T_{\tilde{\mathbf{r}}(t)} M \), \( \forall t \).

Define \( \tilde{e}_i(t) = \tilde{y}_i(e_i(t)) \).

Write \( \tilde{\mathbf{J}}(t) = \sum_{i=1}^n y_i e_i(t) \).

we have \( \tilde{\mathbf{J}}(t) = \sum_{i=1}^n y_i \tilde{e}_i(t) \)

(because \( \tilde{y}_i \) is linear at \( t \)).

\[
\frac{D^3 \tilde{\mathbf{r}}}{dt^3} = \sum_{i=1}^n y_i'' e_i(t)
\]

and

\[
\frac{D^2 \tilde{\mathbf{r}}}{dt^2} = \sum_{i=1}^n y_i'' \tilde{e}_i(t)
\]
The Jacobi equation is equivalent to

\[ \dot{v}_i : 0 = y''_i + \langle R(\dot{y}(t), J(t)) \dot{y}(t), e_i(t) \rangle \]

\[ = y''_i + \langle \dot{R}(\dot{y}(t), \dot{J}(t)) \dot{y}(t), e_i(t) \rangle \]

So \( \tilde{J} \) is a Jacobi field.

So \( \tilde{J}(t) = d(\exp_{t \tilde{y}'(0)} (t \frac{DJ(0)}{dt})) \)

\[ = \psi_t(J(t)) \]

\[ = \psi_t(d(\exp_{t \tilde{y}'(0)} (t \frac{DJ(0)}{dt}))) \]

\( \psi_0 = i \), so

\[ \frac{DJ}{dt}(0) = \psi_0 \left(\frac{DJ}{dt}(0)\right) \]

Also \( \tilde{y}'(0) = i \tilde{x}(0) = i \left(\frac{DJ}{dt}(0)\right) \)

Therefore:

\[ d(\exp_{t \tilde{y}'(0)} (t \frac{DJ}{dt}(0))) = \]

\[ = \psi_t(d(\exp_{t \tilde{y}'(0)})) \left( t \frac{DJ}{dt}(0) \right) \]
So \( y_t = d(\exp_{t \cdot i \varphi'(0)}) \circ d(\exp^{-1}_{t \cdot \varphi'(0)}) \) (take \( \frac{dJ}{dt}(0) = d(\exp_{t \cdot \varphi'(0)})^{-1}(\text{something}) \)).

Now plug \( u \mid t=0 \) to get \( df_p = i \).

**Corollaries:**

Given two manifolds \( M, \tilde{M} \) of the same dimension and equal constant curvature, \( \forall \varphi \in T_p M, \tilde{\varphi} \in T_{\tilde{p}} \tilde{M} \).
and any isometry $i : T_p M \to T_{\tilde{p}} \tilde{M}$, 

exists a neighborhood $V$ of $\tilde{p}$ and a neighborhood $W$ of $\tilde{p}$ and an isometry $f : V \to W$ s.t. $df_p = i$.

2) Same result, but take $M = \tilde{M}$, $p = \tilde{p}$: this means that manifolds with constant curvature have a lot of local isometries.

Hyperbolic $n$-space: $H^n$ as a differentiable manifold:

$H^n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0 \}$.

We endow this with the metric $g = \sum_{i=1}^{n} \frac{1}{x^2_i} dx_i^2$.

At any point $a = (a_1, \ldots, a_n) \in H^n$, we can
identify: $T_a H^n$ with $\mathbb{R}^n$.

then $g_{ij} = \frac{\delta_{ij}}{a^n}$

$(d\xi_i)(d\xi^j) = g_{ij} = \delta_{ij}$

Claim: $H^n$ is a complete Riemannian manifold.

We prove that geodesics are defined on all of $\mathbb{R}$.

Claim: The geodesics are the vertical half lines and the vertical half circles with center on $\{x_n = 0\}$.

First we prove that there are geodesics: