Variations of energy:

Definition: Suppose $c : [0,a] \to M$ is a piecewise differentiable curve. A variation of c is a continuous map

$f : (-\varepsilon, \varepsilon) \times [0,a] \longrightarrow M$

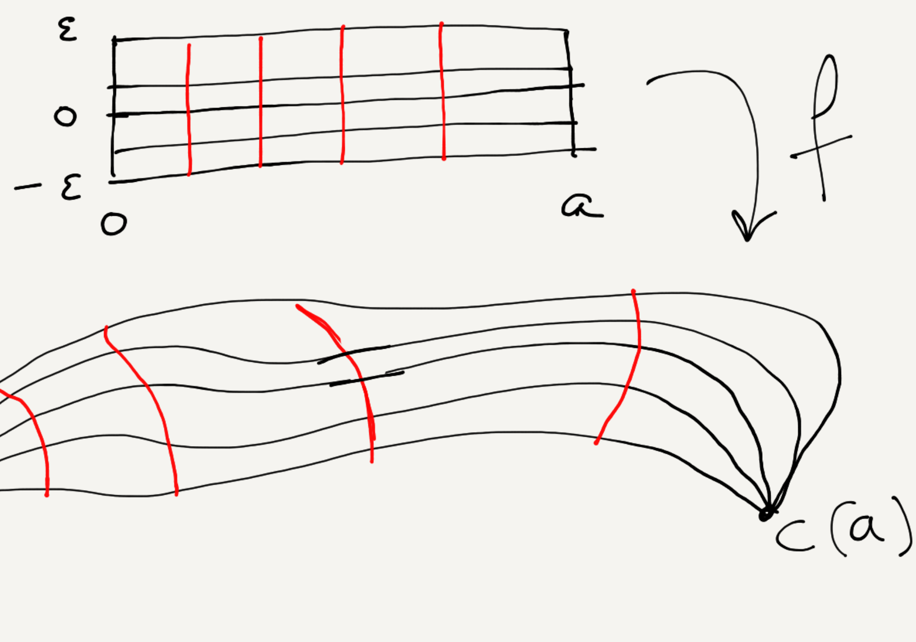
s.t. (a) $f(0,t) = c(t) \quad \forall t \in [0,a]$

(b) $\exists \; t_0 = 0, t_1, \dots, t_k, t_{k+1} = a$

s.t. f is differentiable on $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$ $\forall i$

f is called proper (a proper variation) if $\forall s \quad f(s,0) = c(0), \; f(s,a) = c(a)$

Picture:



The curves $f_t(s)$ for fixed t are the transverse curves of the variation. The variational field of f is the vector field $V(t) := \frac{\partial f}{\partial s}(0,t)$ on c.

Proposition 1: Given a piecewise differentiable vector field $V(t)$ along the piecewise differentiable curve c, $\exists$ a variation f of c s.t. V is the variational field of f. If $V(0) = V(a) = 0$, f can be chosen to be proper.

Proof: $\forall\ t \in [0,a]$, $\exists\ \delta_t > 0$, $\exists$ totally normal neighborhood W_t of $c(t)$ s.t. $\forall\ q \in W_t$, $W_t \subset \exp_q(B_{\delta_t}(0))$

$W_t \cap c([0,a]) \supset c([t', t''])$ for some $(t', t'') \ni t$.

We can cover $c([0,a])$ with a finite number of $c([t',t''])$ as above. Let $\delta := \min\{\delta_t\}$ for these finitely many t.

$$N := \max_{t\in[0,a]} \{|V(t)|\}$$

Now define $f(s,t) := \exp_{c(t)}(sV(t))$ this is well-defined for $s \in (-\varepsilon, \varepsilon)$ with $\varepsilon = \dfrac{\delta}{N}$.

Let us verify that f is a variation with variational field V.

f is piecewise differentiable because $\exp$ is differentiable and V is piecewise differentiable.

$$\left.\frac{\partial f}{\partial s}\right|_{s=0} = \frac{\partial}{\partial s}\left(\exp_{c(t)}(sV(t))\right) = d\left(\exp_{c(t)}\right)_{sV(t)}(V(t))$$
$$= d(\exp_{c(t)})_0 (V(t)) = V(t)$$

If $V(0) = V(a) = 0$, then

$$f(s,t) = \exp_{c(t)}(sV(t))$$

and $f(s,0) = \exp_{c(0)}(sV(0))$

$$= \exp_{c(0)}(0) = c(0)$$

similarly $f(s,a) = c(a) \quad \forall s \quad \square$

<u>Definition:</u> The energy function of a variation $f: (-\varepsilon, \varepsilon) \times [0,a] \longrightarrow M$ is the function $E: (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ defined by $E(s) := \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right|^2 dt$

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the length of the curve $f_s(t) = f(s,t)$ is $L(s) := \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right| dt$

Given two piecewise continuous non negative functions g, h on $[0,a]$, $\langle g, h \rangle := \int_0^a gh \, dt$ defines an inner product on the space of such functions. So we have a Schwarz inequality

$$\left(\int_0^a g(t)\,h(t)\,dt\right)^2 \leq \int_0^a g(t)^2\,dt \int_0^a h(t)^2\,dt$$

apply this to $g(t)=1$, $h(t)=\left|\frac{\partial f}{\partial t}(s,t)\right|$ to obtain $L(s)^2 \leq a\,E(s)$

<u>Lemma 2</u>: Suppose given $p, q \in M$ and let $\gamma : [0,a] \longrightarrow M$ be a length minimizing geodesic from p to q. Then for any piecewise differentiable curve $c : [0,a] \longrightarrow M$ s.t. $c(0)=p$, $c(a)=q$, we have

$$E(\gamma) \leq E(c)$$

with equality if and only if c is a minimizing geodesic.

Proof: Since γ is a geodesic, $\frac{d\gamma}{dt}$ has constant length, say l, so

$$E(\gamma) = \int_0^a \left|\frac{d\gamma}{dt}\right|^2 dt = a l^2$$

$$L(\gamma) = \int_0^a \left|\frac{d\gamma}{dt}\right| dt = a l$$

and $L(\gamma)^2 = a E(\gamma)$

Hence $a E(\gamma) = L(\gamma)^2 \leq L(c)^2 \leq a E(c)$

$$\Rightarrow \quad E(\gamma) \leq E(c)$$

If we have equality iff we have equality in the Schwarz inequality iff the two functions $1, \left|\frac{dc}{dt}\right|$ are proportional (except possibly at finitely many points). So $\frac{dc}{dt}$ has constant length, and $L(c) = L(\gamma)$ by result from first quarter (3.9 in Chap. 3) c is a geodesic. $\square$

Next we write a formula for

$$\frac{dE}{ds}; \qquad E(s) = \int_0^a \left|\frac{\partial f}{\partial t}(s,t)\right|^2 dt$$

$$E(s) = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t) \right\rangle dt .$$

$$\frac{dE}{ds} = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{d}{ds} \left\langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t) \right\rangle dt$$

$$= \sum_{i=0}^{k} 2 \int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial t}(s,t), \frac{D}{ds}\frac{\partial f}{\partial t}(s,t) \right\rangle dt$$

$$= \sum_{i=0}^{k} 2 \int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial t}(s,t), \frac{D}{dt}\frac{\partial f}{\partial s}(s,t) \right\rangle dt$$

$$= \sum_{i=0}^{k} \left\{ 2 \int_{t_i}^{t_{i+1}} \frac{d}{dt} \left\langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial s}(s,t) \right\rangle dt - 2 \int_{t_i}^{t_{i+1}} \left\langle \frac{D}{dt}\frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial s}(s,t) \right\rangle dt \right\}$$

$$= \sum_{i=0}^{k} 2 \left(\left\langle \frac{\partial f}{\partial t}(s,t_{i+1}), \frac{\partial f}{\partial s}(s,t_{i+1}) \right\rangle - \left\langle \frac{\partial f}{\partial t}(s,t_i), \frac{\partial f}{\partial s}(s,t_i) \right\rangle \right)$$

$$- 2 \int_0^a \left\langle \frac{D}{dt}\frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial s}(s,t) \right\rangle dt$$

Now set $s=0$ and $V(t)=\frac{\partial f}{\partial s}(0,t)$

$f(0,t)=c(t)$

we obtain:

$$\frac{dE}{ds}\Big|_{s=0} = \sum_{i=0}^{k} 2\left(\left\langle \frac{dc}{dt}(t_{i+1}), V(t_{i+1})\right\rangle - \left\langle \frac{dc}{dt}(t_i), V(t_i)\right\rangle\right)$$

$$-2\int_0^a \left\langle \frac{D}{dt}\frac{dc}{dt}, V(t)\right\rangle dt$$