Now set $s=0$ and $V(t)=\frac{\partial f}{\partial s}(0,t)$

$f(0,t)=c(t)$

we obtain:

$$\frac{dE}{ds}\Big|_{s=0}=\sum_{i=0}^{k} 2\left(\left\langle \frac{dc}{dt}(t_{i+1}^-), V(t_{i+1}^-)\right\rangle - \left\langle \frac{dc}{dt}(t_i^+), V(t_i^+)\right\rangle\right)$$

$$-2\int_0^a \left\langle \frac{D}{dt}\frac{dc}{dt}, V(t)\right\rangle dt$$

$$= -2\left\langle \frac{dc}{dt}(0), V(0)\right\rangle + 2\left\langle \frac{dc}{dt}(t_1^-)-\frac{dc}{dt}(t_1^+), V(t_1)\right\rangle$$

$$+\cdots+2\left\langle \frac{dc}{dt}(t_k^-)-\frac{dc}{dt}(t_k^+), V(t_k)\right\rangle$$

$$+2\left\langle \frac{dc}{dt}(a), V(a)\right\rangle - 2\int_0^a \left\langle \frac{D}{dt}\frac{dc}{dt}, V(t)\right\rangle dt$$

where $\frac{dc}{dt}(t_i^-)=\lim_{t \xrightarrow{<} t_i}\frac{dc}{dt}(t)$

$$\frac{dc}{dt}(t_i^+)=\lim_{t\xrightarrow{>} t_i}\frac{dc}{dt}(t)$$

First application:

Proposition: A piecewise differentiable curve $c : [0,a] \to M$ is a geodesic iff for every proper variation f of c,

$$\frac{dE}{ds}(0) = 0$$

Proof: E is a differentiable function of s with a minimum at 0 if c is a geodesic. So, if c is a geodesic,

$$\frac{dE}{ds}(0) = 0.$$

Conversely, suppose that $\frac{dE}{ds}(0) = 0$ for all proper variations of c.

Choose a piecewise differentiable function $g : [0,a] \longrightarrow \mathbb{R}$ s.t. $g(t_i) = 0 \ \forall i$ and $g(t) > 0$ if $t \neq t_i \ \forall i$.

Put $V(t) := g(t) \frac{D}{dt} \frac{dc}{dt}$

We will show that $V(t)=0 \ \forall t$.

Let $f(s,t) \left(= \exp_{c(t)}(sV(t))\right)$ be a variation with variational field V. Then

$$0 = \frac{dE}{ds}(0) = -2\int_0^a g(t) \left\langle \frac{D}{dt}\frac{dc}{dt}, \frac{D}{dt}\frac{dc}{dt} \right\rangle dt$$

by the previous formula.

Since $g(t)>0$ for $t\neq t_i$ and $\left\langle \frac{D}{dt}\frac{dc}{dt}, \frac{D}{dt}\frac{dc}{dt} \right\rangle$ is nonnegative, we deduce $\frac{D}{dt}\frac{dc}{dt}=0$ for all $t\neq t_i$. So c is a geodesic on all the intervals (t_i, t_{i+1}).

Now choose a differentiable vector field V along c s.t. $V(0)=V(a)=0$ and, for $t_i\neq 0, a$,

$$V(t_i) = \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-).$$

Using again the previous formula and the fact that c is a geodesic on (t_i, t_{i+1}),

$$0 = \frac{dE}{ds}(0) = -2\sum_{i=1}^{k} \left\langle \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) \right\rangle$$

Therefore $\frac{dc}{dt}(t_i^+) = \frac{dc}{dt}(t_i^-) \; \forall i$ and c is C^1. $\frac{D}{dt}\frac{dc}{dt} = 0$ so c satisfies the geodesic equations which are ordinary differential equations. By the uniqueness and existence of solutions of ODEs, c is C^∞, hence a geodesic. $\square$

We will see next that if the Ricci curvature of a Riemannian manifold is bounded from below by a positive number, then the manifold is compact and we will obtain a bound on its diameter (the largest distance between two points).

For this we will need the second derivative of the energy function for which we now derive a formula.

As before $f: (-\varepsilon, \varepsilon) \times [0,a]$ is a variation of $c: [0,a] \to M$ and E is its energy function. Recall the formula:

$$\frac{dE}{ds} = 2 \sum_{i=0}^{k} \left(\left\langle \frac{\partial f}{\partial s}(s, t_{i+1}^-), \frac{\partial f}{\partial t}(s, t_{i+1}^-) \right\rangle - \left\langle \frac{\partial f}{\partial s}(s, t_i^+), \frac{\partial f}{\partial t}(s, t_i^+) \right\rangle \right)$$

$$- 2\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt$$

take the derivative:

$$\frac{d^2E}{ds^2} = 2 \Bigg\{ \sum_{i=0}^{k} \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(s, t_{i+1}^-), \frac{\partial f}{\partial t}(s, t_{i+1}^-) \right\rangle + \left\langle \frac{\partial f}{\partial s}(s, t_{i+1}^-), \frac{D}{ds}\frac{\partial f}{\partial t}(s, t_{i+1}^-) \right\rangle$$

$$- \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(s, t_i^+), \frac{\partial f}{\partial t}(s, t_i^+) \right\rangle$$

$$- \left\langle \frac{\partial f}{\partial s}(s, t_i^+), \frac{D}{ds}\frac{\partial f}{\partial t}(s, t_i^+) \right\rangle$$

$$- \int_0^a \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt$$

$$- \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt \Bigg\}$$

Now put $s=0$, then

$$f(0,t) = \gamma(t) \quad \text{is a geodesic}$$

and $f(s,0) = \gamma(0)$, $f(s,a) = \gamma(a)$
$\forall\, s$ because f is proper.

So $$\frac{D}{dt}\frac{\partial f}{\partial t}\Big|_{s=0} = \frac{D}{dt}\frac{d\gamma}{dt} = 0$$

and

$$\sum_{i=0}^{k} \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,t_{i+1}^{-}), \frac{\partial f}{\partial t}(0,t_{i+1}^{-})\right\rangle - \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,t_i^{+}), \frac{\partial f}{\partial t}(0,t_i^{+})\right\rangle$$

$$= \sum_{i=0}^{k} \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,t_{i+1}^{-}), \frac{\partial f}{\partial t}(0,t_{i+1})\right\rangle - \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,t_i^{+}), \frac{\partial f}{\partial t}(0,t_i)\right\rangle$$

$= 0$ <u>Claim</u>: next time.

So

$$\frac{d^2E}{ds^2}(0) = 2\left\{\sum_{i=0}^{k}\left\langle \frac{\partial f}{\partial s}(0,t_{i+1}^-), \frac{D}{ds}\frac{\partial f}{\partial t}(0,t_{i+1}^-)\right\rangle - \left\langle \frac{\partial f}{\partial s}(0,t_i^+), \frac{D}{ds}\frac{\partial f}{\partial t}(0,t_i^+)\right\rangle - \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt\right\}$$

$$= 2\left\{\sum_{i=0}^{k}\left\langle V(t_{i+1}^-), \frac{D}{dt}V(t_{i+1}^-)\right\rangle - \left\langle V(t_i^+), \frac{D}{dt}V(t_i^+)\right\rangle - \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt\right\}$$

$$= 2\left\{\sum_{i=1}^{k}\left\langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+)\right\rangle - \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt.\right\}$$

Use Lemma 4.1 of Chapter 4:

$$\frac{D}{ds}\frac{D}{dt}\left(\frac{\partial f}{\partial t}\right) - \frac{D}{dt}\frac{D}{ds}\left(\frac{\partial f}{\partial t}\right) = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$$

So $\displaystyle\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt$

$$= \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} \right\rangle dt + \int_0^a \left\langle \frac{\partial f}{\partial s}, R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} \right\rangle dt$$

$$= \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} \right\rangle dt + \int_0^a \left\langle \frac{\partial f}{\partial s}, R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} \right\rangle dt$$

at $s=0$

$$= \int_0^a \left\langle V(t), \frac{D^2}{dt^2} V(t) \right\rangle dt + \int_0^a \left\langle V(t), R\left(\frac{d\gamma}{dt}, V(t)\right)\frac{d\gamma}{dt} \right\rangle dt$$

In summary, we have:

$$\frac{d^2E}{ds^2}(0) = 2\Big\{ \sum_{i=1}^{k} \left\langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+) \right\rangle - \int_0^a \left\langle V(t), \frac{D^2}{dt^2}V(t) \right\rangle dt - \int_0^a \left\langle V(t), R\left(\frac{d\gamma}{dt}, V(t)\right)\frac{d\gamma}{dt} \right\rangle dt \Big\}$$

This is the formula we will use.