In summary, we have:

$$\frac{d^2E}{ds^2}(0) = 2\left\{ \sum_{i=1}^{k} \left\langle V(t_i), \frac{D}{dt}V(t_i^-) - \frac{D}{dt}V(t_i^+) \right\rangle \right.$$
$$- \int_0^a \left\langle V(t), \frac{D^2}{dt^2} V(t) \right\rangle dt$$
$$\left. - \int_0^a \left\langle V(t), R\left(\frac{d\gamma}{dt}, V(t)\right)\frac{d\gamma}{dt} \right\rangle dt \right\}$$

This is the formula we will use.

Our first application is the theorem of Bonnet and Myers: (3.1 in Chapter 9)

<u>Theorem:</u> Suppose M is a complete Riemannian manifold and $\exists\, r > 0$ s.t. $\forall p \in M$ $v \in T_pM$ with $|v| = 1$

$$\mathrm{Ric}_p(v) \geq \frac{1}{r^2}.$$

Then M is compact with diameter $\leq \pi r$.

<u>Proof</u>: By the Hopf-Rinow theorem, if we prove that $\operatorname{diam}(M) \leq \pi r$ it will follow that M is compact.
We will show that $\forall\, p, q \in M, d(p,q) \leq \pi r$

Let $p, q \in M$ be arbitrary points.
By the results of Chapter 7, since M is complete, $\exists$ minimizing geodesic γ from p to q. $\gamma : [0,1] \longrightarrow M$

$$\gamma(0) = p \;, \gamma(1) = q, l := l(\gamma) = |\gamma'(t)|.$$

Show $l \leq \pi r$.

Choose parallel vector fields $e_1(t), \dots, e_{n-1}(t)$ along γ in such a way that $\forall\, t$ $\left\{ e_1(t), \dots, e_{n-1}(t), \frac{\gamma'(t)}{l} \right\}$ is an orthonormal basis of $T_{\gamma(t)} M$.

Define $V_i(t) := \sin(\pi t)\, e_i(t)$
$i = 1, \dots, n-1$.

Each V_i can be written as the variation field of a proper variation with energy function $E_i(s)$ and

$$\frac{d^2 E_i}{ds^2}(0) = -2\int_0^1 \left\langle V_i(t), \frac{D^2 V_i}{dt^2} + R\left(\frac{d\gamma}{dt}, V_i\right)\frac{d\gamma}{dt}\right\rangle dt$$

the e_i are parallel, so:

$$\frac{D^2 V_i}{dt^2} = \frac{D^2}{dt^2}\left(\sin(\pi t)\, e_i\right) = -\pi^2 \sin(\pi t)\, e_i$$

recall $e_n(t) = \frac{1}{l}\gamma'(t)$ and

$$\mathrm{Ric}_{\gamma(t)}(e_n(t)) = \frac{1}{n-1}\sum_{i=1}^{n-1} \langle R(e_n, e_i)e_n, e_i\rangle$$

We have

$$\frac{d^2 E_i}{dt^2}(0) = +2\int_0^1 \langle \sin(\pi t)e_i, \pi^2 \sin(\pi t)e_i - R(\gamma', \sin(\pi t)e_i)\gamma'\rangle dt$$

$$= 2\int_0^1 \sin^2(\pi t)\langle e_i, \pi^2 e_i - l^2 R(e_n, e_i)e_n\rangle dt$$

$$= 2\int_0^1 \sin^2(\pi t)\left(\pi^2 - l^2\langle e_i, R(e_n, e_i)e_n\rangle\right) dt$$

$$\sum_{i=1}^{n-1} \frac{d^2 E_i}{ds^2}(0) =$$

$$= 2\int_0^1 \sin^2(\pi t)\left((n-1)\pi^2 - l^2 \sum_{i=1}^{n-1} \langle e_i, R(e_n, e_i)e_n\rangle\right)dt$$

$$= 2\int_0^1 \sin^2(\pi t)\left((n-1)\pi^2 - (n-1)l^2 \mathrm{Ric}_{\gamma(t)}(e_n(t))\right)dt$$

assumption: $\mathrm{Ric}_{\gamma(t)}(e_n(t)) \geq \frac{1}{r^2}$, so:

$$\leq 2\int_0^1 \sin^2(\pi t)(n-1)\left(\pi^2 - \frac{l^2}{r^2}\right)dt$$

$$= 2(n-1)\left(\pi^2 - \frac{l^2}{r^2}\right)\int_0^1 \sin^2(\pi t)\, dt$$

Since γ minimizes energy, we have

$$\frac{d^2 E_i}{ds^2}(0) \geq 0, \text{ so.}$$

$$\pi^2 - \frac{l^2}{r^2} \geq 0 \Rightarrow l \leq \pi r.$$

$\square$

Recall that the universal cover of M can be endowed with a Riemannian structure in such a way that the covering map $p: \tilde{M} \to M$ is a local isometry. So if $Ric_p(v) \geq \frac{1}{r^2}$ on M, the inequality also holds on $\tilde{M}$. So $\tilde{M}$ is compact of diameter $\leq \pi r$. Then $\forall\ p \in M$

$$p^{-1}(p) = \text{closed in } \tilde{M}$$
$$\Rightarrow \text{compact and discrete}$$
$$\Rightarrow \text{finite.}$$

So $\tilde{M}$ is a finite cover of M.

$\Rightarrow$ $\boxed{M \text{ has finite fundamental group}}$ because the fundamental group is in bijection with the fibers of p.

Another application is the Weinstein and Synge theorem: (3.7 in Chapter 9)

<u>Theorem!</u> Let f be an isometry of a compact oriented Riemannian manifold with positive sectional curvature. Suppose

that f preserves the orientation if n is even and reverses it if n is odd.
<u>Then f</u> has a fixed point.

For the proof we need the following:

<u>Lemma:</u> Suppose A is an orthogonal linear transformation of $\mathbb{R}^{n-1}$ with determinant $\det A = (-1)^n$. Then A admits 1 as an eigenvalue.

<u>Proof:</u> The complex eigenvalues of A have modulus 1, their product is $(-1)^n$ and the imaginary ones form pairs of conjugate complex numbers. The product of these pairs of conjugate complex numbers is 1.
So the product of the real eigenvalues is $(-1)^n = \det A$.
So, if n is even, A has at least one real eigenvalue $= 1$.
If n is odd, $\det A = -1$ so we have