that f preserves the orientation if n is even and reverses it if n is odd. Then f has a fixed point.

For the proof we need the following:

Lemma: Suppose A is an orthogonal linear transformation of $\mathbb{R}^{n-1}$ with determinant $\det A = (-1)^n$. Then A admits 1 as an eigenvalue.

Proof: The complex eigenvalues of A have modulus 1, their product is $(-1)^n$ and the imaginary ones form pairs of conjugate complex numbers. The product of these pairs of conjugate complex numbers is 1.
So the product of the real eigenvalues is $(-1)^n = \det A$.
So, if n is even, A has at least one real eigenvalue $= 1$.
If n is odd, $\det A = -1$ so we have

an even number of real eigenvalues whose product is -1. So they cannot all be -1: at least one of them has to be 1. $\square$.

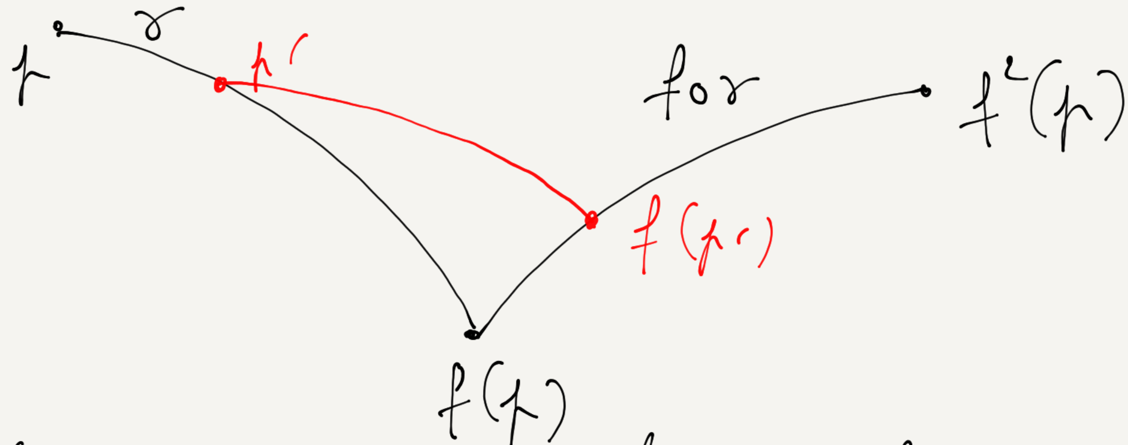
Proof of the theorem of Synge & Weinstein:

Assume f does not have a fixed point. Then $d(p, f(p)) > 0 \quad \forall\, p \in M$.

M is compact and $d(p, f(p))$ is a continuous function on M. So $d(p, f(p))$ has a minimum on M. Let $p \in M$ be a point where the minimum is reached. $\quad l := d(p, f(p))$

Let $\gamma : [0, l] \longrightarrow M$ be a minimizing normalized geodesic with $\gamma(0) = p$, $\gamma(l) = f(p)$.

The strategy of the proof is to construct a variation of γ on which $\frac{d^2E}{ds^2}(0) < 0$ which implies γ is a maximum of $E(s)$ and gives a contradiction.



<u>Claim:</u> the curve formed by concatenating γ and $f\circ\gamma$ is a geodesic.

For this we show that it minimizes length which will follow if we show that the triangle inequality is an equality.

Choose an arbitrary point $p' = \gamma(t')$

We have the triangle inequality:

$$d(p', f(p')) \leq d(p', f(p)) + d(f(p), f(p'))$$

f is an isometry $= d(p', f(p)) + d(p, p')$

we are on a geodesic $= d(p, f(p)) = l$

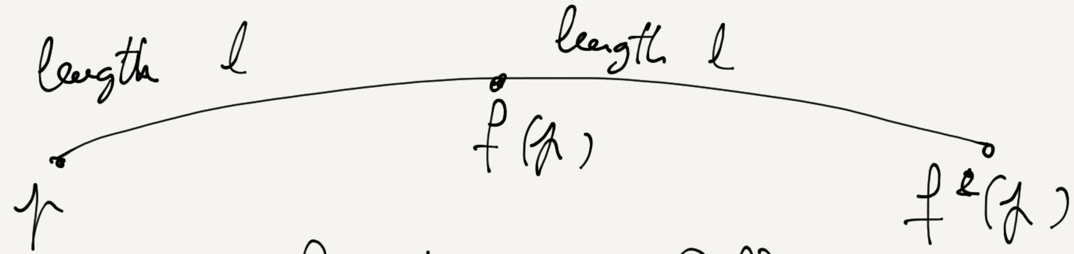
$$= \operatorname{Min}\{d(q, f(q)) \mid q \in M\}$$

$$\Rightarrow d(p', f(p')) = d(p, f(p)) = l$$

$$\Rightarrow d(p', f(p')) = d(p', f(p)) + d(f(p), f(p'))$$

$\Rightarrow$ Claim.

So:



$$\Rightarrow \gamma'(l) = (f\circ\gamma)'(0) = (df)_p(\gamma'(0))$$

$$(df)_p : T_pM \longrightarrow T_{f(p)}M \cong \mathbb{R}^n$$

let $P : T_{f(p)}M \longrightarrow T_pM$ be parallel transport along γ and put

$$\tilde{A} = P\circ(df)_p : T_pM \longrightarrow T_pM$$

$\tilde{A}$ is an orthogonal transformation and

$$\tilde{A}(\gamma'(0)) = P((df)_p(\gamma'(0))) = P(\gamma'(l))$$
$$= \gamma'(0)$$

So $\tilde{A}$ induces $A : \gamma'(0)^\perp \longrightarrow \gamma'(0)^\perp$

$$\gamma'(0)^\perp \cong \mathbb{R}^{n-1}$$

A is an orthogonal transformation and by the hypothesis on f and because P preserves orientation, we have

$$\det A = \det \tilde{A} = \det (df)_p = (-1)^n$$

We apply the lemma to conclude that 1 is

an eigenvalue of $A : \gamma'(0)^\perp \to \gamma'(0)^\perp$
$$\gamma'(0)^\perp \subset T_p M$$

Let $e_1(0)$ be a unit eigenvector for 1 and parallel transport $e_1(0)$ along γ to obtain a parallel unit length vector field along γ which is always orthogonal to γ'.

Now we construct a variation with variation field e_1:

$$h(s,t) := \exp_{\gamma(t)}(s\, e_1(t))$$

We have:

$$\frac{d^2E}{ds^2} = 2\Big\{ \Big\langle \frac{D}{ds}\frac{\partial h}{\partial s}(s,l), \frac{\partial h}{\partial t}(s,l)\Big\rangle + \Big\langle \frac{\partial h}{\partial s}(s,l), \frac{D}{ds}\frac{\partial h}{\partial t}(s,l)\Big\rangle - \Big\langle \frac{D}{ds}\frac{\partial h}{\partial s}(s,0), \frac{\partial h}{\partial t}(s,0)\Big\rangle - \Big\langle \frac{\partial h}{\partial s}(s,0), \frac{D}{ds}\frac{\partial h}{\partial t}(s,0)\Big\rangle - \int_0^l \Big\langle \frac{\partial h}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial h}{\partial t}\Big\rangle dt \Big\}$$

We manipulate the integral as before to obtain

$$\int_0^l \left\langle e_1(t), \frac{D^2}{dt^2} e_1(t) \right\rangle dt$$

$$+ \int_0^l \left\langle e_1(t), R\left(\frac{d\gamma}{dt}, e_1(t)\right)\frac{d\gamma}{dt} \right\rangle dt$$

the first integral is 0 because $e_1(t)$ is parallel.

For the other terms we have:

$$\left\langle \frac{D}{ds}\frac{\partial h}{\partial s}(0,l), \frac{\partial h}{\partial t}(0,l) \right\rangle + \left\langle \frac{\partial h}{\partial s}(0,l), \frac{D}{ds}\frac{\partial h}{\partial t}(0,l) \right\rangle$$

$$- \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}(0,0), \frac{\partial h}{\partial t}(0,0) \right\rangle - \left\langle \frac{\partial h}{\partial s}(0,0), \frac{D}{ds}\frac{\partial h}{\partial t}(0,0) \right\rangle$$

$$= \frac{d}{ds}\left\langle \frac{\partial h}{\partial s}(s,l), \frac{\partial h}{\partial t}(s,l) \right\rangle \Big|_{s=0}$$

$$- \frac{d}{ds}\left\langle \frac{\partial h}{\partial s}(s,0), \frac{\partial h}{\partial t}(s,0) \right\rangle \Big|_{s=0}$$

Since $\frac{\partial h}{\partial t}(0,t) = \gamma'(t)$, $\frac{\partial h}{\partial s}(0,t) = e_1(t)$ and $e_1(t) \perp \gamma'(t)$, the above is 0, So $$\frac{d^2E}{ds^2}(0) = -2\int_0^\ell \langle e_1, R(\gamma', e_1)\gamma'\rangle\, dt$$

$$= -2\int_0^\ell K(e_1, \gamma')\, dt$$

< 0 because $K > 0$ everywhere.

$\square$