So $\frac{d^2 E}{ds^2}(0) = -2\int_0^l \langle e_1, R(\gamma', e_1)\gamma' \rangle dt$

$$= -2\int_0^l K(e_1, \gamma')\, dt$$

< 0 because $K > 0$ everywhere.

$\square$

<u>Corollary</u> of the Synge-Weinstein thm:

If M is a compact manifold with positive sectional curvature, then

1) if M is orientable with even dim, then M is simply connected.

2) if $\dim M$ is odd, then M is orientable.

<u>Note</u>: The hypotheses are necessary:

$\mathbb{P}^2\mathbb{R} \left(= S^2/\{\pm 1\}\right)$ not orientable even dim.

not simply connected.

So orientability is necessary in 1).

and odd dim is necessary in 2).

$\mathbb{P}^3\mathbb{R}^3 \left(= S^3/\{\pm 1\}\right)$ orientable, compact not simply connected

so even dim. is necessary in 1).

<u>Proof of the corollary:</u>

1) Let $\tilde{M} \xrightarrow{\pi} M$ be the universal cover of M. We can put a Riemannian structure on $\tilde{M}$ in such a way that π is a local isometry. M is compact with positive curvature, so $K^M \geq \delta > 0$ on M. (take $\delta = \min\{K\}$)

Since π is a local isometry, we also have $K^{\tilde{M}} \geq \delta > 0$ on $\tilde{M}$.

Apply the Bonnet-Myers theorem to obtain that $\tilde{M}$ is compact. ($\Rightarrow \tilde{M}$ is a finite cover of M).

We show that $\tilde{M} \to M$ has no covering transformations other than $Id_{\tilde{M}}$.

Fact: a covering transformation has no fixed points unless it is the identity.

Let $f: \tilde{M} \to \tilde{M}$ be a covering transformation. M is orientable and even dimensional $\Rightarrow$ $\tilde{M}$ is also orientable and even-dimensional

f is an isometry because the Riemannian structure on $\tilde{M}$ is lifted from M. We can orient M, then lift the orientation to $\tilde{M}$: this implies that f preserves the orientation of $\tilde{M}$.

Synge - Weinstein $\Rightarrow$ f has a fixed pt

$\Rightarrow$ $f = Id$.

2) If M is not orientable, let M' be the orientable double cover. Lift the Riemannian structure of M to M'.

Let $f: M' \to M'$ be a covering transf..

As before f is an isometry.
f is orientation reversing since otherwise M would be orientable.

Synge-Weinstein $\Rightarrow$ f has a fixed pt
$\Rightarrow f = Id$.

$\square$

The Rauch comparison theorem:

Let M be a Riemannian manifold and let $\gamma : [0,a] \longrightarrow M$ be a geodesic. Fix $t_0 \in [0,a]$. On the vector space of piecewise differentiable vector fields along γ, define the symmetric bilinear form:

$$I_{t_0}(V,W) := \int_0^{t_0} \left\{ \left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle - \langle R(\gamma',V)\gamma', W \rangle \right\} dt$$

the associated quadratic form is:

$$\int_0^{t_0} \left\{ \left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle - \langle R(\gamma',V)\gamma', V \rangle \right\} dt .$$

Index lemma: Suppose that there are no points conjugate to $\gamma(0)$ in the image of γ. Let J be a Jacobi field along γ with $\langle J, \gamma' \rangle = 0 \ \forall t$. Let V be a piecewise differentiable vector field along γ with $\langle V, \gamma' \rangle = 0 \ \forall t$. If $J(0) = V(0) = 0$ and $\exists t_0 \in (0,a]$ s.t. $J(t_0) = V(t_0)$, then

$$I_{t_0}(J, J) \leq I_{t_0}(V, V)$$

with equality iff $J = V$ on $[0, t_0]$.

Proof: The space of Jacobi fields with $J(0) = 0$ and $\langle J, \gamma' \rangle = 0$ has dim. $n-1$. Let $\{J_1, \dots, J_{n-1}\}$ be a basis of this space.

$\exists$ constants $a_1, \dots, a_{n-1}$ s.t.

$$J = \sum_{i=1}^{n} a_i J_i$$