Lemma 5.5 of Chapter 0:

Let $h : (-\delta, \delta) \times U \longrightarrow \mathbb{R}$ be a differentiable map where $\delta > 0$ and $U \subset M$ is open. Suppose $h(0,q) = 0$ $\forall q \in U$. Then $\exists$ a differentiable map $g : (-\delta, \delta) \times U \longrightarrow \mathbb{R}$ s.t. $h(t,q) = t g(t,q)$ $\forall (t,q) \in (-\delta,\delta) \times U$.

In particular, $g(0,q) = \frac{\partial h}{\partial t}(0,q)$.

We will apply the lemma with $U = \phi$.

We can apply the lemma to $J_1, \dots, J_{n-1}$:

If $t \neq 0$, we can just define $A_i = \frac{1}{t} J_i$

First extend the geodesic to

$(-\delta, a] \longrightarrow M$

then restrict to $(-\delta, \delta)$

Choose δ small enough so that the image of $(-\delta, \delta)$ is contained in a coordinate neighborhood $U_{\gamma(0)}$ of $\gamma(0)$.

Then, we can write $J_i = \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial x_j}$

apply Lemma 5.5 to λ_j $\forall i$ to obtain

$J_i = t A_i$ with A_i a differentiable vector field along γ s.t.

$$\left.\frac{DJ_i}{dt}\right|_{t=0} = A_i(0).$$

So $A_1(0), \dots, A_{n-1}(0)$ are linearly independent. Furthermore $J_i(t) \perp \gamma'(t)$ $\forall i$. So $A_i(t) \perp \gamma'(t)$ $\forall t \neq 0$.

$\Rightarrow A_i(t) \perp \gamma'(t)$ $\forall t$ by continuity.

So $\{A_1(t), \dots, A_{n-1}(t)\}$ form a basis of $\gamma'(t)^\perp$ $\forall t$ and we can write

$$V(t) = \sum_{i=1}^{n-1} g_i(t) A_i(t)$$ where

$g_i(t)$ are piecewise differentiable.

$V(0) = 0 \Rightarrow g_i(0) = 0 \quad \forall i$

We can apply Lemma 5.5 of Chapter 0 to $g_1, \dots, g_{n-1}$ to obtain $f_1(t), \dots, f_{n-1}(t)$ s.t. $g_i(t) = t f_i(t)$ and $g_i'(0) = f_i(0)$.

So $\quad V(t) = \sum_{i=1}^{n-1} t f_i(t) A_i(t) \quad \forall t$

$$V(t) = \sum_{i=1}^{n-1} f_i(t) J_i(t) \quad \forall t \in [0,a]$$

Now:

$$\left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle - \langle R(\gamma', V)\gamma', V \rangle =$$

$$\left\langle \sum_{i=1}^{n-1} f_i' J_i + \sum_{i=1}^{n-1} f_i \frac{DJ_i}{dt}, \sum_{j=1}^{n-1} f_j' J_j + \sum_{j=1}^{n-1} f_j \frac{DJ_j}{dt} \right\rangle$$

$$- \left\langle \sum_{i=1}^{n-1} f_i R(\gamma', J_i)\gamma', \sum_{j=1}^{n-1} f_j J_j \right\rangle$$

$$= \sum_{i,j} f_i' f_j' \langle J_i, J_j \rangle + \sum_{i,j} f_i f_j \left\langle \frac{DJ_i}{dt}, \frac{DJ_j}{dt} \right\rangle$$

$$+ \sum_{i,j} f_i f_j' \left\langle \frac{DJ_i}{dt}, J_j \right\rangle + \sum_{i,j} f_i' f_j \left\langle J_i, \frac{DJ_j}{dt} \right\rangle$$

$$+ \sum_{i,j} f_i f_j \left\langle \frac{D^2 J_i}{dt^2}, J_j \right\rangle$$

We are going to write the last line as part of a derivative. Since

$$f_i \frac{D^2 J_i}{dt^2} = \frac{D}{dt}\left(f_i \frac{DJ_i}{dt} \right) - f_i' \frac{DJ_i}{dt},$$

the only possibility is the derivative

$$\frac{d}{dt}\left\langle \sum f_i \frac{DJ_i}{dt}, \sum f_j J_j \right\rangle$$

Now we compute this derivative and compare with the previous sum.

$$= \left\langle \frac{D}{dt} \sum f_i \frac{DJ_i}{dt}, \sum f_j J_j \right\rangle + \left\langle \sum f_i \frac{DJ_i}{dt}, \frac{D}{dt} \sum f_j J_j \right\rangle$$

$$= \left\langle \sum f_i' \frac{DJ_i}{dt} + \sum f_i \frac{D^2 J_i}{dt^2}, \sum f_j J_j \right\rangle$$

$$+ \left\langle \sum f_i \frac{DJ_i}{dt}, \sum f_j' J_j + \sum f_j \frac{DJ_j}{dt} \right\rangle$$

These all appear in the previous sum except $\left\langle \sum f_i' \frac{DJ_i}{dt}, \sum f_j J_j \right\rangle$

instead we have $\left\langle \sum f_i' J_i, \sum f_j \frac{DJ_j}{dt} \right\rangle$

the difference is

$$\sum f_i' f_j \left(\left\langle \frac{DJ_i}{dt}, J_j \right\rangle - \left\langle J_i, \frac{DJ_j}{dt} \right\rangle \right)$$

Put $h_{ij}(t) := \left\langle \frac{DJ_i}{dt}, J_j \right\rangle - \left\langle J_i, \frac{DJ_j}{dt} \right\rangle$

we have $h_{ij}(0) = 0$

$$h_{ij}' = \frac{d}{dt}\left(\left\langle \frac{DJ_i}{dt}, J_j \right\rangle - \left\langle J_i, \frac{DJ_j}{dt} \right\rangle\right)$$

$$= \left\langle \frac{D^2 J_i}{dt^2}, J_j \right\rangle - \left\langle J_i, \frac{D^2 J_j}{dt^2} \right\rangle$$

$$= -\langle R(\gamma', J_i)\gamma', J_j \rangle + \langle J_i, R(\gamma', J_j)\gamma' \rangle$$

by symmetry.

$\Rightarrow h_{ij}$ is constant

$\Rightarrow h_{ij} \equiv 0$ because $h_{ij}(0) = 0$

So $\left\langle \frac{DJ_i}{dt}, J_j \right\rangle = \left\langle J_i, \frac{DJ_j}{dt} \right\rangle \ \forall i, j$

So $\left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle - \langle R(\gamma', V)\gamma', V \rangle =$

$$= \frac{d}{dt}\left\langle \sum f_i \frac{DJ_i}{dt}, \sum f_j J_j \right\rangle + \sum f_i' f_j' \langle J_i, J_j \rangle$$

and

$$I_{t_0}(V, V) = \left\langle \sum f_i \frac{DJ_i}{dt}, \sum f_j J_j \right\rangle (t_0) + \int_0^{t_0} \left\langle \sum f_i' J_i, \sum f_j' J_j \right\rangle dt$$

$$I_{t_0}(J, J) = \left\langle \sum a_i \frac{DJ_i}{dt}(t_0), \sum a_j J_j(t_0) \right\rangle$$

$a_i = f_i(t_0)$ because $V(t_0) = J(t_0)$

So $I_{t_0}(V,V) - I_{t_0}(J,J) =$

$$= \int_0^{t_0} \langle \sum f_i' J_i, \sum f_j' J_j \rangle \, dt$$

$$= \int_0^{t_0} \left| \sum f_i' J_i \right|^2 dt \geq 0$$

This is 0 iff $\sum f_i' J_i \equiv 0$

$\Rightarrow f_i' \equiv 0 \; \forall i \Rightarrow f_i$ is constant $\forall i$

$f_i(t) = f_i(t_0) = a_i$

$\Rightarrow V = J$ on $[0, t_0]$ $\square$

Rauch's theorem: Suppose M and $\tilde{M}$ are Riemannian manifolds of dim. n & $n+k$ resp. Suppose given a geodesic $\gamma : [0,a] \to M$ and a geodesic $\tilde{\gamma} : [0,a] \to \tilde{M}$ s.t. $|\gamma'(t)| = |\tilde{\gamma}'(t)|$.

Let J and $\tilde{J}$ be Jacobi fields along γ & $\tilde{\gamma}$ resp. s.t. $J(0) = \tilde{J}(0) = 0$, $\left|\frac{DJ}{dt}(0)\right| = \left|\frac{D\tilde{J}}{dt}(0)\right|$

and $\langle \frac{DJ}{dt}(0), \gamma'(0) \rangle = \langle \frac{D\tilde{J}}{dt}(0), \tilde{\gamma}'(0) \rangle$.

Assume that $\tilde{\gamma}$ has no conjugate points on $(0,a]$ and $\forall t, \forall x \in T_{\gamma(t)}M$, $\tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$

$$\tilde{K}(\tilde{x}, \tilde{\gamma}'(t)) \geq K(x, \gamma'(t))$$

where K and $\tilde{K}$ are the sectional