So $I_{t_0}(V,V) - I_{t_0}(J,J) =$

$$= \int_0^{t_0} \langle \sum f_i' J_i, \sum f_j' J_j \rangle dt$$

$$= \int_0^{t_0} \left|\sum f_i' J_i\right|^2 dt \geq 0$$

This is 0 iff $\sum f_i' J_i \equiv 0$

$\Rightarrow f_i' \equiv 0 \ \forall i \Rightarrow f_i$ is constant $\forall i$

$f_i(t) = f_i(t_0) = a_i$

$\Rightarrow V = J$ on $[0, t_0]$ $\square$

Rauch's theorem: Suppose M and $\tilde{M}$ are Riemannian manifolds of dim. n & $n+k$ resp. Suppose given a geodesic $\gamma : [0,a] \to M$ and a geodesic $\tilde{\gamma} : [0,a] \to \tilde{M}$ s.t. $|\gamma'(t)| = |\tilde{\gamma}'(t)|$. Let J and $\tilde{J}$ be Jacobi fields along γ & $\tilde{\gamma}$ resp. s.t. $J(0) = \tilde{J}(0) = 0$, $\left|\frac{DJ}{dt}(0)\right| = \left|\frac{D\tilde{J}}{dt}(0)\right|$

and $\langle \frac{DJ}{dt}(0), \gamma'(0) \rangle = \langle \frac{D\tilde{J}}{dt}(0), \tilde{\gamma}'(0) \rangle$.

Assume that $\tilde{\gamma}$ has no conjugate points on $(0,a]$ and $\forall t, \forall x \in T_{\gamma(t)}M$, $\tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$

$$\tilde{K}(\tilde{x}, \tilde{\gamma}'(t)) \geq K(x, \gamma'(t))$$

where K and $\tilde{K}$ are the sectional curvatures of M and $\tilde{M}$ resp..

Then $\quad |\tilde{J}| \leq |J|$.

If $\exists\ t_0 \in (0, a]$ s.t. $\quad |\tilde{J}(t_0)| = |J(t_0)|$,

then $\quad \tilde{K}(\tilde{J}(t), \tilde{\gamma}'(t)) = K(J(t), \gamma'(t))$

$\forall \quad t \in [0, t_0]$.

<u>Proof</u>: Recall Proposition 3.6 of Chapter 5:

If J is a Jacobi field along $\gamma : [0, a] \to M$, then

$$\langle J(t), \gamma'(t)\rangle = \langle \frac{DJ}{dt}(0), \gamma'(0)\rangle t + \langle J(0), \gamma'(0)\rangle$$

Apply this to J and $\tilde{J}$ to obtain:

$$\langle J(t), \gamma'(t)\rangle = \langle \tilde{J}(t), \tilde{\gamma}'(t)\rangle \quad \forall t$$

$$\left(= \langle \frac{DJ}{dt}(0), \gamma'(0)\rangle t \right)$$

$$\frac{\langle J(t), \gamma'(t)\rangle}{|\gamma'(t)|^2} \gamma'(t)$$ is the projection of $J(t)$ onto $\gamma'(t)$. We replace J

with $\quad J(t) - \dfrac{\langle J(t), \gamma'(t)\rangle}{\langle \gamma'(t), \gamma'(t)\rangle} \gamma'(t)$

and $\tilde{J}$ with $\tilde{J}(t) - \dfrac{\langle \tilde{J}(t), \tilde{\gamma}'(t)\rangle}{\langle \tilde{\gamma}'(t), \tilde{\gamma}'(t)\rangle} \tilde{\gamma}'(t)$.

So we can assume $\langle J(t), \gamma'(t) \rangle = 0$ and $\langle \tilde{J}(t), \tilde{\gamma}'(t) \rangle = 0 \quad \forall t$.

If $\frac{DJ}{dt}(0) = 0$, then $J(t) = 0 \ \forall t$.

So assume $\frac{DJ}{dt}(0) \neq 0$ (hence $\frac{D\tilde{J}}{dt}(0) \neq 0$)

We are going to compare $v(t) := |J(t)|^2$ and $\tilde{v}(t) := |\tilde{J}(t)|^2$. Since $\tilde{\gamma}$ has no conjugate points, $\frac{v(t)}{\tilde{v}(t)}$ is well-defined on $(0, a]$. By l'Hôpital:

$$\lim_{t \to 0^+} \frac{v(t)}{\tilde{v}(t)} = \lim_{t \to 0^+} \frac{v'(t)}{\tilde{v}'(t)} = \lim_{t \to 0^+} \frac{v''(t)}{\tilde{v}''(t)}$$

$$v'(t) = \langle J(t), J(t) \rangle' = 2 \left\langle \frac{DJ}{dt}, J \right\rangle$$

$$v''(t) = 2 \left\langle \frac{D^2 J}{dt^2}, J \right\rangle + 2 \left\langle \frac{DJ}{dt}, \frac{DJ}{dt} \right\rangle$$

$$\Rightarrow \lim_{t \to 0^+} \frac{v(t)}{\tilde{v}(t)} = \frac{\left| \frac{DJ}{dt}(0) \right|^2}{\left| \frac{D\tilde{J}}{dt}(0) \right|^2} = 1$$

If we show that $\frac{v}{\tilde{v}}$ is an increasing function, it will follow that $\frac{v}{\tilde{v}} \geq 1 \ \forall t$.

This means $\left(\frac{v}{\tilde{v}} \right)' \geq 0 \ \forall t$, or $v' \tilde{v} - v \tilde{v}' \geq 0$.

Fix $t_0 \in (0, a]$. Want to show:

$$r'(t_0)\tilde{r}(t_0) - r(t_0)\tilde{r}'(t_0) \geq 0$$

If $r(t_0) = 0$, then $J(t_0) = 0$ and $r'(t_0) = 0$ and the inequality is trivially satisfied. So we can assume $r(t_0) \neq 0$.

$\tilde{r}(t_0) \neq 0$ because γ has no conjugate points.

So we prove $\dfrac{r'(t_0)}{r(t_0)} \geq \dfrac{\tilde{r}'(t_0)}{\tilde{r}(t_0)}$.

$$\frac{r'(t_0)}{r(t_0)} = \frac{2\langle \frac{DJ}{dt}(t_0), J(t_0)\rangle}{\langle J(t_0), J(t_0)\rangle}$$

<u>Note</u>: $r'(t_0) = \displaystyle\int_0^{t_0} r''(t)\,dt$

$$= 2\int_0^{t_0}\left(\left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle + \left\langle \frac{D^2J}{dt^2}, J\right\rangle\right)dt$$

$$= 2\int_0^{t_0}\left(\left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle - \langle R(\gamma', J)\gamma', J\rangle\right)dt$$

$$= 2\, I_{t_0}(J, J)$$

So $\dfrac{v'(t_0)}{v(t_0)} = \dfrac{2}{v(t_0)} I_{t_0}(J, J)$

$$= 2 I_{t_0}\left(\frac{J}{\sqrt{v(t_0)}}, \frac{J}{\sqrt{v(t_0)}}\right)$$

similarly: $\dfrac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} = 2\tilde{I}_{t_0}\left(\dfrac{\tilde{J}}{\sqrt{\tilde{v}(t_0)}}, \dfrac{\tilde{J}}{\sqrt{\tilde{v}(t_0)}}\right)$

To compare the above, we need vector fields on the same manifold.

Choose $e_1(t), \dots, e_n(t)$ to be parallel vector fields on M s.t. $\{e_1(t), \dots, e_n(t)\}$ is an orthonormal basis of $T_{\gamma(t)}M, \forall t$.

Similarly choose $\tilde{e}_1(t), \dots, \tilde{e}_{n+k}(t)$ on $\tilde{M}$.

Choose $e_1(t) = \dfrac{\gamma'(t)}{|\gamma'(t)|}$, $\tilde{e}_1(t) = \dfrac{\tilde{\gamma}'(t)}{|\tilde{\gamma}'(t)|}$

Since we want to apply the index lemma, we also assume $e_2(t_0) = \dfrac{J(t_0)}{|J(t_0)|}$

$\tilde{e}_2(t_0) = \dfrac{\tilde{J}(t_0)}{|\tilde{J}(t_0)|}$.

Any vector field V along γ can be written as $V(t) = \sum_{i=1}^{n} g_i(t)\, e_i(t)$

define $\varphi(V) := \sum_{i=1}^{n} g_i(t)\tilde{e}_i(t)$

Then $\langle \varphi(V_1), \varphi(V_2) \rangle_{\tilde{M}} = \langle V_1, V_2 \rangle_M$

and $\dfrac{D(\varphi(V))}{dt} = \varphi\left(\dfrac{DV}{dt}\right)$

because $\{e_i(t)\}$ and $\{\tilde{e}_i(t)\}$ are orthonormal bases of parallel vector fields.

Put $U := \dfrac{J}{|J(t_0)|}$ and $\tilde{U} = \dfrac{J}{|\tilde{J}(t_0)|}$

By the index lemma:

$$I_{t_0}(\varphi(U), \varphi(U)) \geqslant I_{t_0}(\tilde{U}, \tilde{U})$$

because $\tilde{U}$ is a Jacobi field.

$$I_{t_0}(\varphi(U), \varphi(U)) = \int_0^{t_0} \left(\left\langle \frac{D}{dt}\varphi(U), \frac{D}{dt}\varphi(U) \right\rangle - \langle \tilde{R}(\tilde{\gamma}', \varphi(U))\tilde{\gamma}', \varphi(U) \rangle \right) dt$$

$$= \int_0^{t_0} \left(\left\langle \varphi\frac{DU}{dt}, \varphi\frac{DU}{dt} \right\rangle - \langle \tilde{R}(\tilde{\gamma}', \varphi(U))\tilde{\gamma}', \varphi(U) \rangle \right) dt$$

$$= \int_0^{t_0} \left(\left\langle \frac{DU}{dt}, \frac{DU}{dt} \right\rangle - \langle \tilde{R}(\tilde{\gamma}', \varphi(U))\tilde{\gamma}', \varphi(U) \rangle \right) dt$$

$$\leq \int_0^{t_0} \left(\left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle - \langle R(\gamma', V)\gamma', V \rangle \right) dt$$

using the hypothesis $\tilde{K} \geq K$ and $|\gamma'| = |\tilde{\gamma}'|$

$$= I_{t_0}(V, V).$$

So $\quad I_{t_0}(V, V) \geq I_{t_0}(\tilde{V}, \tilde{V})$

$$\begin{array}{ccc} I_{t_0}(V,V) & \geq & I_{t_0}(\tilde{V},\tilde{V}) \\ \parallel & & \parallel \\ \frac{1}{2}\,\frac{r'(t_0)}{r(t_0)} & & \frac{1}{2}\,\frac{\tilde{r}'(t_0)}{\tilde{r}(t_0)} \end{array}$$