$$\leq \int_0^{t_0} \left(\left\langle \frac{DU}{dt}, \frac{DU}{dt} \right\rangle - \langle R(\gamma', U)\gamma', U \rangle \right) dt$$

using the hypothesis $\tilde{K} \geq K$ and

$|\gamma'| = |\tilde{\gamma}'|$

$$= I_{t_0}(U, U).$$

So

$$\begin{array}{ccc} I_{t_0}(U,U) & \geq & I_{t_0}(\tilde{U}, \tilde{U}) \\ \| & & \| \\ \frac{1}{2}\,\frac{v'(t_0)}{v(t_0)} & & \frac{1}{2}\,\frac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} \end{array}$$

Second part of proof:

Assume $\exists\, t_0 \in (0, a]$ s.t.

$$|\tilde{J}(t_0)| = |J(t_0)|$$

$$\iff v(t_0) = \tilde{v}(t_0) \iff \frac{v(t_0)}{\tilde{v}(t_0)} = 1$$

Since $\frac{v}{\tilde{v}}$ is an increasing function with limit 1 as $t \to 0$, $\frac{v(t)}{\tilde{v}(t)} = 1 \ \forall\, t \leq t_0$

and $\left(\frac{v}{\tilde{v}}\right)'(t) = 0 \quad \forall\, t \in (0, t_0)$

So $(r'\tilde{r} - r\tilde{r}')(t) = 0 \quad \forall\, t \in [0,t_0]$

and $\dfrac{r'}{r} = \dfrac{\tilde{r}'}{\tilde{r}} \quad \forall\, t \in (0,t_0]$

$$\Rightarrow \quad I_{t_0}(J,J) = I_{t_0}(\tilde{J},\tilde{J})$$
$$= I_{t_0}(\varphi(J),\varphi(J))$$

$$\Rightarrow \int_0^{t_0} \left(\left\langle \frac{DJ}{dt}, \frac{DJ}{dt} \right\rangle - \langle R(\gamma',J)\gamma',J\rangle \right) dt$$

$$= \int_0^{t_0} \left(\left\langle \frac{D\tilde{J}}{dt}, \frac{D\tilde{J}}{dt} \right\rangle - \langle \tilde{R}(\tilde{\gamma}',\tilde{J})\tilde{\gamma}',\tilde{J}\rangle \right) dt$$

$$= \int_0^{t_0} \left(\left\langle \varphi\left(\frac{DJ}{dt}\right), \varphi\left(\frac{DJ}{dt}\right) \right\rangle - \langle \tilde{R}(\tilde{\gamma}',\varphi(J))\tilde{\gamma}',\varphi(J)\rangle \right) dt$$

$$\Rightarrow \int_0^{t_0} \left(\langle R(\gamma',J)\gamma',J\rangle - \langle \tilde{R}(\tilde{\gamma}',\tilde{J})\tilde{\gamma}',\tilde{J}\rangle \right) dt = 0$$

By assumption the function in the integral is ≤ 0 everywhere. Since its integral is 0, the function is identically 0.

$$\Rightarrow \quad K(\gamma',J) = K(\tilde{\gamma}',\tilde{J}) \text{ on } [0,t_0]$$

□

We can use Rauch's theorem to estimate the distance between conjugate points:

Proposition (2.4 in Chapter 10):
Suppose that the sectional curvature K of M always satisfies $K \leq H$ (resp. $L \leq K$) for some positive constant H (resp. L). Then $\forall$ geodesics γ on M, the distance d between two conjugate points on γ is $d \geq \frac{\pi}{\sqrt{H}}$ (resp. $d \leq \frac{\pi}{\sqrt{L}}$).

Proof: Compare M with the sphere S^n_H of radius $\frac{1}{\sqrt{H}}$. The distance between two conjugate points on S is $\pi \frac{1}{\sqrt{H}}$.
Let $\gamma : [0, a] \rightarrow M$ be a geodesic, J a Jacobi field along γ s.t. $J(0) = 0$ and $\langle J, \gamma' \rangle \equiv 0$. Let $\tilde{\gamma} : [0, a] \rightarrow S^n_H$ be a geodesic, $\tilde{J}$ a Jacobi field along $\tilde{\gamma}$ with $\tilde{J}(0) = 0$ $\langle \tilde{J}, \tilde{\gamma}' \rangle \equiv 0$ and

$$\left|\frac{DJ}{dt}(0)\right| = \left|\frac{D\tilde{J}}{dt}(0)\right|.$$

Assume furthermore that $|\gamma'| = |\tilde{\gamma}'| = 1$ and $a < \frac{\pi}{\sqrt{H}}$ so that there are no conjugate points on $\tilde{\gamma}$.

Then $\underbrace{|J(t)| \geq |\tilde{J}(t)|}_{>0}$ $\forall\, t \in [0,a]$

$\Rightarrow$ $|J(t)| > 0$ $\forall\, t \in [0,a]$

true for any Jacobi field J

$\Rightarrow$ $\nexists$ conjugate points on γ

$\Rightarrow$ $d \geq a$ $\forall\, a < \frac{\pi}{\sqrt{H}}$

$\Rightarrow$ $d \geq \frac{\pi}{\sqrt{H}}$

Now assume: $L \leq K$ and compare M with the sphere S^n_L of radius $\frac{1}{\sqrt{L}}$.

Do the same argument with $M = S^n_L$, $\tilde{M} = M$. Do it by contradiction, assuming $d > \frac{\pi}{\sqrt{L}}$. $\square$

We can also compare lengths of curves.

<u>Proposition</u> (2.5 in Chapter 10)

Let M and $\tilde M$ be Riemannian manifolds of dim n. Suppose that $\tilde K \geq K$ always. Let $p \in M$, $\tilde p \in \tilde M$, $i : T_pM \to T_{\tilde p}\tilde M$ an isometry. Choose $r > 0$ s.t.

$\exp_p|_{B_r(0)}$ is a diffeomorphism and

$\exp_{\tilde p}|_{B_r(0)}$ is a local diffeo. (i.e., $d(\exp_p)$ is always injective).

Let $c : [0, a] \to B_r(p) = \exp_p(B_r(0))$ be a diff. curve and put

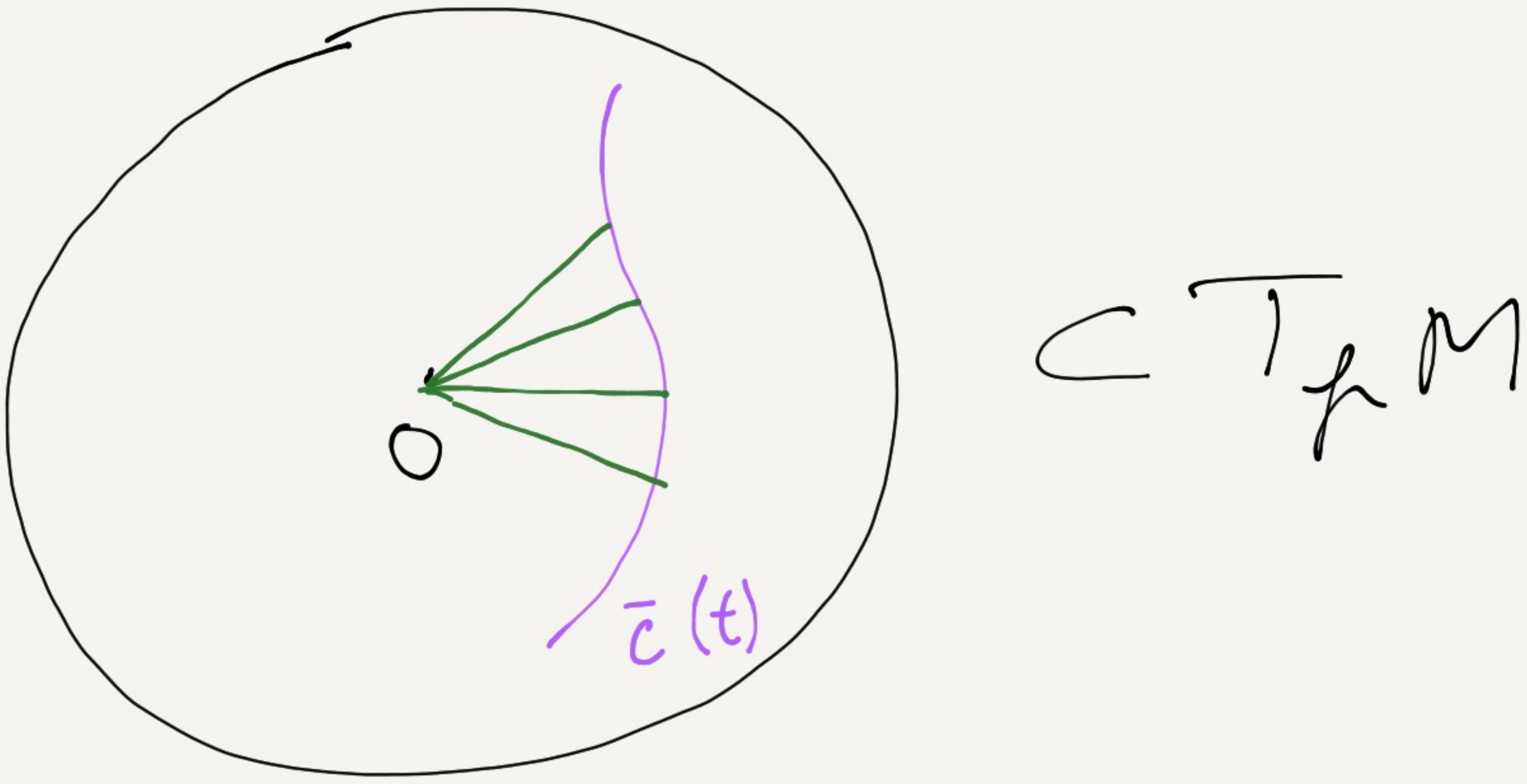
$$\tilde c := \exp_{\tilde p} \circ i \circ \exp_p^{-1} \circ c : [0,a] \to \tilde M$$

$$\begin{tikzcd}
 & T_pM \arrow[r, "i"] \arrow[d, "\exp_p"] & T_{\tilde p}\tilde M \arrow[d, "\exp_{\tilde p}"] \\
c : [0,a] \arrow[ur, "\bar c"] \arrow[r] & M & \tilde M
\end{tikzcd}$$

.

Then $l(\tilde{c}) \leq l(c)$.

Proof:



$\bar{c} := \exp_p^{-1} \circ c : [0,a) \longrightarrow B_r(0) \subset T_pM$

Define $f(s,t) := \exp_p(t\bar{c}(s))$

Then, as we saw in Chapter 5,

$J(t) := \frac{\partial f}{\partial s}(0,t)$ is a Jacobi field

with $J(0) = 0$, $\frac{DJ}{dt}(0) = \bar{c}'(0)$

Similarly $J_s(t) := \frac{\partial f}{\partial s}(s,t)$ fixed s

is a Jacobi field with $J_s(0) = 0$

and $\frac{DJ_s}{dt}(0) = \bar{c}'(s)$.

Similarly, define $\tilde{f}(s,t) := \exp_{\tilde{p}}(t(i \circ \bar{c}(s)))$

and we have the Jacobi fields

$\tilde{J}_s(t) := \frac{\partial \tilde{f}}{\partial s}(s,t)$

with $\tilde{J}_s(0) = 0$, $\frac{D\tilde{J}_s}{dt}(0) = (i \circ \bar{c})'(s)$.

Then $\left|\frac{DJ_s}{dt}(0)\right| = |\bar{c}'(s)| = |(i\circ\bar{c})'(s)| = \left|\frac{D\tilde{J}_s}{dt}(0)\right|$

$J_s(t)$ is a Jacobi field along the geodesic $\gamma_s(t) = \exp_p(t\,\bar{c}(s))$ and $\tilde{J}_s(t)$ is a Jacobi field along the geodesic $\tilde{\gamma}_s(t) = \exp_{\tilde{p}}(t(i\circ\bar{c})(s))$

$$\left\langle \frac{DJ_s}{dt}(0), \gamma_s'(0)\right\rangle = \langle \bar{c}'(s), \gamma_s'(0)\rangle$$
$$= \langle i\circ\bar{c}'(s), i\circ\gamma_s'(0)\rangle$$

because i is linear and an isometry

$$= \left\langle \frac{D\tilde{J}_s}{dt}(0), \tilde{\gamma}_s'(0)\right\rangle$$

Now apply Rauch's theorem to obtain

$$|\tilde{J}_s| \le |J_s|$$

Note: $J_s(t) = \frac{\partial f}{\partial s}(s,t) = d(\exp_p)_{t\bar{c}(s)}(t\bar{c}'(s))$

$\Rightarrow \; J_s(1) = d(\exp_p)_{\bar{c}(s)}(\bar{c}'(s)) = c'(s)$

because $c(s) = \exp_p(\bar{c}(s))$