\[ \leq \int_{t_0}^{\infty} \left( \langle \frac{\partial U}{\partial t}, \frac{\partial U}{\partial t} \rangle - \langle R(\tilde{x}', \tilde{u}) \tilde{x}', \tilde{u} \rangle \right) dt \]

using the hypothesis \( K \geq K' \) and \( |\tilde{z}'| = |\tilde{z}| \)

\[ = I_{t_0} (v, u). \]

So \( I_{t_0} (v, u) \geq I_{t_0} (\tilde{v}, \tilde{u}) \)

\[ \frac{1}{2} \frac{\tilde{v}'(t_0)}{\sqrt{v(t_0)}} \leq \frac{1}{2} \frac{\tilde{v}'(t_0)}{\sqrt{v(t_0)}} \]

Second part of proof:

Assume \( \exists t_0 \in (0, a) \) s.t.

\[ | \tilde{v}(t_0) | = | \tilde{v}(t_0) | \]

\[ \Rightarrow \tilde{v}(t_0) = \tilde{v}(t_0) \Rightarrow \frac{v(t_0)}{\sqrt{v(t_0)}} = 1 \]

Since \( \frac{v}{\sqrt{v}} \) is an increasing function with

\[ \lim_{t \to 0} \frac{v(t)}{\sqrt{v(t)}} = 1 \forall t \leq t_0 \]

and \( \left( \frac{v}{\sqrt{v}} \right)'(t) = 0 \forall t \in (0, t_0) \)
\[ (\tilde{\gamma}', \tilde{\gamma}' - \tilde{\gamma}' \tilde{\gamma}') (t) = 0 \quad \forall \ t \in [0, t_0] \]

and \[ \frac{\tilde{\gamma}'}{\tilde{\gamma}} = \frac{\tilde{\gamma}}{\tilde{\gamma}} \quad \forall \ t \in (0, t_0) \]

\[ \Rightarrow \quad I_{t_0} (J, J) = I_{t_0} (\tilde{J}, \tilde{J}) = I_{t_0} (\varphi (J), \varphi (J)) \]

\[ \Rightarrow \quad \int_{0}^{t_0} \left( \left< \frac{DJ}{dt} , \frac{DJ}{dt} \right> - \left< R(\gamma', J) \gamma', J \right> \right) dt \]

\[ = \int_{0}^{t_0} \left( \left< \frac{d\tilde{\gamma}}{dt} , \frac{d\tilde{\gamma}}{dt} \right> - \left< \tilde{R}(\tilde{\gamma}', \tilde{J}) \tilde{\gamma}', \tilde{J} \right> \right) dt \]

\[ = \int_{0}^{t_0} \left( \left< \varphi (\frac{DJ}{dt}) , \varphi (\frac{DJ}{dt}) \right> - \left< \tilde{R}(\tilde{\gamma}', \varphi (J)) \tilde{\gamma}', \varphi (J) \right> \right) dt \]

\[ \Rightarrow \quad \int_{0}^{t_0} \left( \left< R(\gamma', J) \gamma', J \right> - \left< \tilde{R}(\tilde{\gamma}', \tilde{J}) \tilde{\gamma}', \tilde{J} \right> \right) dt = 0 \]

By assumption, the function in the integral is \( \leq 0 \) everywhere. Since its integral is 0, the function is identically 0.

\[ \Rightarrow \quad K(\gamma', J) = K(\tilde{\gamma}', \tilde{J}) \quad \text{on} \ [0, t_0] \]
We can use Munch's theorem to estimate the distance between conjugate points:

**Proposition (2.4 in Chapter 10):**

Suppose that the sectional curvature \( K \) of \( M \) always satisfies \( K \leq H \) (resp. \( L \leq K \)) for some positive constant \( H \) (resp. \( L \)). Then a geodesic \( \gamma \) on \( M \), the distance \( d \) between two conjugate points on \( \gamma \) is \( d \geq \frac{\pi}{\sqrt{H}} \) (resp. \( d \leq \frac{\pi}{\sqrt{L}} \)).

**Proof:** Compare \( M \) with the sphere \( S^m \) of radius \( \frac{1}{\sqrt{H}} \). The distance between two conjugate points on \( S \) is \( \pi \frac{1}{\sqrt{H}} \).

Let \( \gamma : [0, a] \to M \) be a geodesic, \( J \) a Jacobi field along \( \gamma \) s.t. \( J(0) = 0 \) and \( \langle J, \gamma' \rangle \equiv 0 \). Let \( \tilde{\gamma} : [0, a] \to S^m \) be a geodesic, \( \tilde{J} \) a Jacobi field along \( \tilde{\gamma} \) with \( \tilde{J}(0) = 0 \) and \( \langle \tilde{J}, \tilde{\gamma}' \rangle \equiv 0 \).
\[ \left| \frac{dJ}{dt} (0) \right| = \left| \frac{d\tilde{J}}{dt} (0) \right|. \]

Assume furthermore that \( |\xi'| = |\tilde{\xi}'| = 1 \)
and \( a < \frac{\pi}{\sqrt{H}} \) so that there are no conjugate points on \( \tilde{\gamma} \).

Then \( |J(t)| \geq |\tilde{J}(t)| \neq 0 \) \( t \in [0, a] \)
\( > 0 \)
\( \Rightarrow |J(t)| > 0 \neq 0 \) \( t \in [0, a] \)
ture for any Jacobi field \( J \)
\( \Rightarrow \) no conjugate points on \( \gamma \)
\( \Rightarrow a \neq a < \frac{\pi}{\sqrt{H}} \)
\( \Rightarrow d > \frac{\pi}{\sqrt{H}} \)

Now assume \( L \leq K \) and compare \( M \) with the sphere \( S^u_L \) of radius \( \frac{1}{\sqrt{L}} \).
Do the same argument with \( M = S^u_L \)
\( \tilde{M} = M \). Do it by contradiction, assuming \( d > \frac{\pi}{\sqrt{L}} \). \( \square \)
We can also compare lengths of curves.

**Proposition (2.5 in Chapter 1)**

Let $M$ and $\tilde{M}$ be Riemannian manifolds of dim $n$. Suppose that $\tilde{K} \geq K$ always.

Let $p \in M$, $\tilde{x} \in \tilde{M}$, $i : T_p M \to T_{\tilde{p}} \tilde{M}$ an isometry. Choose $\kappa > 0$ s.t.

$\exp_p |_{B_\kappa (0)}$ is a diffeomorphism and

$\exp_{\tilde{p}} |_{B_\kappa (0)}$ (i.e., $d(\exp_p)$ is always) injective.

Let $c : [0, a] \to B_\kappa (p) = \exp_p (B_\kappa (0))$ be a diff. curve and put

$\tilde{c} := \exp_{\tilde{p}} \circ i \circ \exp_p \circ c : [0, a] \to \tilde{M}$

$c : [0, a] \to M \quad \tilde{c} : [0, a] \to \tilde{M}$
Then \( l(\tau) \leq l(c) \).

**Proof:**

\[ \tilde{c} := \exp_p^{-1} \circ c : [0, a] \to B_n(0) \subset T_p M \]

Define \( f(s, t) := \exp_p (t \tilde{c}(s)) \)

Then, as we saw in Chapter,

\[ J(t) := \frac{\partial f}{\partial s}(0, t) \] is a Jacobi field

with \( J(0) = 0 \), \( \frac{DJ}{dt}(0) = \tilde{c}'(0) \)

Similarly, \( J_s(t) := \frac{\partial f}{\partial s}(s, t) \) fixed \( s \)

is a Jacobi field with \( J_s(0) = 0 \)

and \( \frac{DJ_s}{dt}(0) = \tilde{c}'(s) \).

Similarly, define \( \tilde{f}(s, t) := \exp_p (t \sigma \tilde{c}(s)) \)

and we have the Jacobi fields

\[ \tilde{J}_s(t) := \frac{\partial \tilde{f}}{\partial s}(s, t) \]
with \[ \tilde{\mathbf{J}}_s(0) = 0, \quad \frac{d\tilde{\mathbf{J}}_s}{dt}(0) = (i0\bar{c})(s). \]

Then \[ \left| \frac{d\mathbf{J}_s}{dt}(0) \right| = |\bar{c}'(s)| = \left| (i0\bar{c})'(s) \right| = \left| \frac{d\tilde{\mathbf{J}}_s}{dt}(0) \right| \]

\( \mathbf{J}_s(t) \) is a Jacobi field along the geodesic
\( \mathbf{c}_s(t) = \exp_t(t\bar{c}(s)) \) and \( \tilde{\mathbf{J}}_s(t) \) is a

Jacobi field along the geodesic
\( \tilde{\mathbf{c}}_s(t) = \exp_t(t(i0\bar{c})(s)) \)

\[ \left< \frac{d\mathbf{J}_s}{dt}(0), \mathbf{c}_s'(0) \right> = \left< \bar{c}'(s), \mathbf{c}_s'(0) \right> \]

\[ = \left< (i0\bar{c}')'(s), (i0\mathbf{c}_s)'(0) \right> \]

\[ = \left< \frac{d\tilde{\mathbf{J}}_s}{dt}(0), \tilde{\mathbf{c}}_s'(0) \right> \]

because \( i \) is linear and an isometry

Now apply Plancherel's theorem to obtain

\[ \left| \tilde{\mathbf{J}}_s \right| \leq \| \mathbf{J}_s \| \]

Note: \[ \mathbf{J}_s(t) = \frac{d}{ds}(s, t) = d(\exp_t)_{\bar{c}(s)}(t\bar{c}'(s)) \]

\[ \Rightarrow \mathbf{J}_s(1) = d(\exp_t)_{\bar{c}(s)}(\bar{c}'(s)) = \mathbf{c}'(s) \]
because \( \mathbf{c}(s) = \exp_t(\bar{c}(s)) \)