with $\tilde{J}_s(0) = 0$, $\frac{D\tilde{J}_s}{dt}(0) = (i \circ \bar{c})'(s)$.

Then $\left|\frac{DJ_s}{dt}(0)\right| = |\bar{c}'(s)| = |(i\circ\bar{c})'(s)| = \left|\frac{D\tilde{J}_s}{dt}(0)\right|$

$J_s(t)$ is a Jacobi field along the geodesic $\gamma_s(t) = \exp_p(t\bar{c}(s))$ and $\tilde{J}_s(t)$ is a Jacobi field along the geodesic $\tilde{\gamma}_s(t) = \exp_{\tilde{p}}(t(i\circ\bar{c})(s))$

$$\left\langle \frac{DJ_s}{dt}(0), \gamma_s'(0)\right\rangle = \langle \bar{c}'(s), \gamma_s'(0)\rangle$$
$$= \langle i\circ\bar{c}'(s), i\circ\gamma_s'(0)\rangle$$ because i is linear and an isometry
$$= \left\langle \frac{D\tilde{J}_s}{dt}(0), \tilde{\gamma}_s'(0)\right\rangle$$

Now apply Rauch's theorem to obtain

$$|\tilde{J}_s| \leq |J_s|$$

Note: $J_s(t) = \frac{\partial f}{\partial s}(s,t) = d(\exp_p)_{t\bar{c}(s)}(t\bar{c}'(s))$

$\Rightarrow J_s(1) = d(\exp_p)_{\bar{c}(s)}(\bar{c}'(s)) = c'(s)$

because $c(s) = \exp_p(\bar{c}(s))$

Similarly: $\tilde{J}_s(1) = \tilde{c}'(s) \ \forall s$

So $\quad |\tilde{c}'(s)| \leq |c'(s)| \quad \forall s$

$$\Rightarrow \ell(\tilde{c}) = \int_0^a |\tilde{c}'(s)|\,dt \leq \int_0^a |c'(s)|\,dt = \ell(c)$$

$\square$

Chapter 11: The Morse index theorem

We want to count the conjugate points along a geodesic.

Let M be a Riemannian manifold and $\gamma : [0,a] \longrightarrow M$ a geodesic.

Denote $\mathcal{V} :=$ the vector space of all piecewise differentiable vector fields V along γ with $V(0) = V(a) = 0$

On $\mathcal{V}$ we define a symmetric bilinear form $I = I_a$ as follows:

$$I(V,W) := \int_0^a \left(\left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle - \langle R(\gamma', V)\gamma', W \rangle \right) dt$$

Definition: The index of I is

$i(I) = \sup\{\dim S \mid S \subset \mathcal{V}$ is of finite dim and $I|_S$ is negative definite$\}$

<u>Definition</u>: The kernel or null space of I is the subspace

$$\operatorname{Ker} I := \{V \in \mathcal{V} \mid I(V,W) = 0 \ \forall W \in \mathcal{V}\}$$

The nullity or corank of I is the dimension of $\operatorname{Ker} I$.

We say I is nondegenerate if $\operatorname{Ker} I = \{0\}$ (degenerate if not).

<u>Remark</u>: If $\mathcal{V}$ had finite dimension and we represent I by a square matrix A, then $\operatorname{Ker} I = \operatorname{Ker} A$.

Our goal in this chapter is to prove

<u>The Morse Index Theorem</u>:

The index of I is finite and equal to the number of points of γ conjugate

to $\gamma(0)$ (other than $\gamma(0), \gamma(a)$)
counted with multiplicities:
If $\gamma(t_1), \dots, \gamma(t_m)$ are the conjugate points with $0 < t_1 < \dots < t_m < a$ with multiplicities $r_1, \dots, r_m$, then the index of I is $\sum_{i=1}^{m} r_i$.

Proposition: $V \in \operatorname{Ker} I \iff V$ is a Jacobi field along γ.

Proof: $I(V,W) = \int_0^a \left(\left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle - \langle R(\gamma', V)\gamma', W \rangle \right) dt$

We have:

$$\frac{d}{dt}\left\langle \frac{DV}{dt}, W \right\rangle = \left\langle \frac{D^2V}{dt^2}, W \right\rangle + \left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle$$

$$\int_0^a \left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle dt = \int_0^a \frac{d}{dt}\left\langle \frac{DV}{dt}, W \right\rangle dt - \int_0^a \left\langle \frac{D^2V}{dt^2}, W \right\rangle dt$$

Let $t_1, \dots, t_k$ be the points where V fails to be differentiable.

$$\int_0^a \left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle dt = \sum_{i=0}^{k} \left[\left\langle \frac{DV}{dt}, W \right\rangle \right]_{t_i}^{t_{i+1}}$$

$t_0 := 0,\ t_{k+1} := a$ $\qquad - \int_0^a \left\langle \frac{D^2 V}{dt^2}, W \right\rangle dt$

$$= - \left\langle \frac{DV}{dt}(t_k^+), W(t_k) \right\rangle + \left\langle \frac{DV}{dt}(t_k^-), W(t_k) \right\rangle$$

$$- \left\langle \frac{DV}{dt}(t_{k-1}^+), W(t_{k-1}) \right\rangle + \cdots + \left\langle \frac{DV}{dt}(t_1^-), W(t_1) \right\rangle$$

$$- \int_0^a \left\langle \frac{D^2 V}{dt^2}, W \right\rangle dt$$

$$= \sum_{i=1}^{k} \left\langle \frac{DV}{dt}(t_i^-) - \frac{DV}{dt}(t_i^+), W(t_i) \right\rangle$$

$$- \int_0^a \left\langle \frac{D^2 V}{dt^2}, W \right\rangle dt$$

Hence $I(V, W) =$

$$\sum_{i=1}^{k} \left\langle \frac{DV}{dt}(t_i^-) - \frac{DV}{dt}(t_i^+), W(t_i) \right\rangle$$

$$+ \int_0^a \left\langle - \frac{D^2 V}{dt^2} - R(\gamma', V)\gamma', W \right\rangle dt$$

So if V is a Jacobi field, $I(V, W) = 0$

$\forall W \in \mathcal{V}$

Now assume $I(V,W)=0 \ \forall W \in \mathcal{V}$.
Let $t_1,\dots,t_k$ again be the points where V fails to be differentiable.
Let $f(t)$ be a differentiable function on $[0,a]$ s.t. $f(t_i)=0 \ \forall i$ and
$f(t)>0 \quad \forall t \neq t_i$.

Choose $W(t) = f(t)\left(\frac{D^2V}{dt^2}+R(\gamma',V)\gamma'\right)$

Then $I(V,W) = -\int_0^a f(t)\left|\frac{D^2V}{dt^2}+R(\gamma',V)\gamma'\right|^2 dt$

$\Rightarrow \quad \frac{D^2V}{dt^2}+R(\gamma',V)\gamma'=0$ away from the t_i

Now choose W to be a vector field s.t.

$W(t_i) = \frac{DV}{dt}(t_i^+) - \frac{DV}{dt}(t_i^-)$, then

$I(V,W) = -\sum_{i=1}^{k}\left|\frac{DV}{dt}(t_i^-) - \frac{DV}{dt}(t_i^+)\right|^2 = 0$

$\Rightarrow \quad \frac{DV}{dt}(t_i^-) = \frac{DV}{dt}(t_i^+) \quad \forall i \Rightarrow V$ is diff.
$\Rightarrow \quad V$ is C^∞ by the uniqueness of solutions of ODE. $\square$