Corollary: $\operatorname{Ker} I \neq \{0\} \iff \gamma(0)$ and $\gamma(a)$ are conjugate points.

Towards the proof of the Morse Index Theorem:

Recall: $\forall\, p \in M$, $\exists\, \delta > 0$ and a neighborhood W of p s.t. $\exp_q$ is a diffeomorphism on $B_\delta(0) \subset T_qM$ $\forall\, q \in W$ and $W \subset \exp_q(B_\delta(0))$.

We called W a totally normal neighborhood of p. (δ depends on p) Denote $W_p := W$ a totally normal neighborhood.

We have $\gamma([0,a]) \subset \bigcup_{t \in [0,a]} W_{\gamma(t)}$.

Assume $W_{\gamma(t)}$ is open $\forall t$.

So $\exists\; t_1 < \dots < t_{k-1} \in (0,a)$ s.t.

$$\gamma([0,a]) \subset W_{t_0} \cup W_{t_1} \cup \dots \cup W_{t_k}$$

where $t_0 = 0$, $t_k = a$ (by compactness of $\gamma([0,a])$).

$\gamma([t_{i-1}, t_i]) \subset W_{\gamma(t)}$ for some t

So $\gamma|_{[t_{i-1}, t_i]}$ is a length minimizing geodesic and $\gamma([t_{i-1}, t_i])$ does not contain any pairs of conjugate points.

<u>Def</u>: A subdivision of $\gamma([0,a])$ as above is called a normal subdivision.

<u>Def</u>: Define $\mathcal{V}^-(t_0, \dots, t_k) =: \mathcal{V}^-$ to be the subvector space of $\mathcal{V} := \mathcal{V}(0,a)$ consisting of vector fields V s.t.

$V|_{[t_{i-1}, t_i]}$ is a Jacobi field $\forall\, i = 1, \dots, k$

Define $\mathcal{V}^+(t_0, \dots, t_k) =: \mathcal{V}^+$ to be the subvector space of $\mathcal{V}$ consisting of vector fields W s.t. $W(t_1) = \dots = W(t_{k-1}) = 0$

<u>Proposition</u>: $\mathcal{V} = \mathcal{V}^- \underset{\bigoplus}{\perp} \mathcal{V}^+$

and $I|_{\mathcal{V}^+}$ is positive definite.

<u>Corollary</u>: The index of I is equal to the index of $I|_{\mathcal{V}^-}$ and the kernel of I

is contained in $\mathcal{V}^-$.

<u>Proof of the proposition:</u>

$V \in \mathcal{V}^0$ $\exists!$ vector field V^- s.t.

$V^-\big|_{[t_{i-1}, t_i]}$ is a Jacobi field $\forall i$ and

$V^{-}(t_i) = V(t_i)$ $\forall i$

Then $V - V^- \in \mathcal{V}^+$

and $V = (V - V^-) + V^- \in \mathcal{V}^+ + \mathcal{V}^-$

So $\mathcal{V} = \mathcal{V}^- + \mathcal{V}^+$

To see that the sum is direct, suppose

$V \in \mathcal{V}^- \cap \mathcal{V}^+$, then $V(t_i) = 0$ $\forall i$

and $V\big|_{[t_{i-1}, t_i]}$ is a Jacobi field. Since

there are no pairs of conjugate points in

$[t_{i-1}, t_i]$, we obtain $V \equiv 0$.

To see that $\mathcal{V}^- \perp \mathcal{V}^+$, let $V \in \mathcal{V}^-$

and $W \in \mathcal{V}^+$ be arbitrary, then

$$I(V,W) = \int_0^a \langle -\frac{D^2V}{dt^2} - R(\gamma', V)\gamma', W\rangle dt$$
$$+ \sum_{i=1}^{k-1} \langle \frac{DV}{dt}(t_i^-) - \frac{DV}{dt}(t_i^+), W(t_i)\rangle$$

$= 0$ because $V \in \mathcal{V}^-$ and $W \in \mathcal{V}^+$.

Now consider $I\big|_{\mathcal{V}^+}$:

Choose $V \in \mathcal{V}^+$, then $I(V,V)$

$$= \int_0^a \left\langle -\frac{D^2V}{dt^2} - R(\gamma', V)\gamma', V \right\rangle dt$$

$$+ \sum_{i=1}^{k-1} \left\langle \frac{DV}{dt}(t_i^-) - \frac{DV}{dt}(t_i^+), V(t_i) \right\rangle$$

$$= \int_0^a -\left\langle \frac{D^2V}{dt^2} + R(\gamma', V)\gamma', V \right\rangle dt$$

$$= \frac{1}{2} \frac{d^2E}{ds^2}(0)$$ the second derivative of the energy of a variation with variation field V.

≥ 0 because γ minimizes energy.

So $I(V,V) \geq 0 \quad \forall\, V \in \mathcal{V}^+$.

Now suppose $I(V,V) = 0$.

Let $W \in \mathcal{V}^+$ be arbitrary, then

$$I(V+cW, V+cW) = 2c\, I(V,W) + c^2 I(W,W)$$

$$\geq 0 \quad \forall\, c \in \mathbb{R}$$

$\Rightarrow$ the discriminant $I(V,W)^2 \leq 0$

$\Rightarrow I(V,W) = 0 \quad \forall\, W \in \mathcal{V}^+$

we also have $I(V,W) = 0 \; \forall\, W \in \mathcal{V}^-$

$\Rightarrow I(V,W) = 0 \quad \forall\, W \in \mathcal{V}$

$\Rightarrow V \in \operatorname{Ker} I$

$\Rightarrow V$ is a Jacobi field $\Rightarrow V \in \mathcal{V}^-$

$\Rightarrow V = 0$ $\square$

<u>Remark:</u> $\mathcal{V}^-$ is finite dimensional.

So <u>Corollary:</u> The index and the nullity of I are finite.

<u>In fact:</u> $\mathcal{V}^- \cong T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_{k-1})}M$

So $\dim \mathcal{V}^- = n(k-1)$

Proof of the Morse Index Theorem:

$\forall\, t \in [0,a]$, we have

$$I_t(V,W) := \int_0^t \left(\left\langle \frac{DV}{d\tau}, \frac{DW}{d\tau} \right\rangle - \langle R(\gamma', V)\gamma', W \rangle \right) d\tau$$

Define $i(t) :=$ index of I_t

We shall see that $i(t)$ is a step function which jumps by the multiplicity at each point conjugate to $\gamma(0)$.

First: If t is small enought, $i(t)=0$

$\gamma([0,t_1])$ contains no pairs of conjugate points

for $t \in [0,t_1]$, $I_t = \int_0^t \cdots$

$$\mathcal{V}(0,t) = \mathcal{V}^-(0,t) \oplus \mathcal{V}^+(0,t)$$

$\Rightarrow \mathcal{V}^-(0,t) = \{0\}$

$\Rightarrow I_t$ is positive definite

$\Rightarrow i(t) = 0$ and $\underset{\substack{\| \\ 0}}{n(t)} :=$ the nullity of I_t

Second: $i(t)$ is non decreasing, meaning

$\forall\ t_1 \le t_2,\quad i(t_1) \le i(t_2)$

$I_{t_1} = \int_0^{t_1} \cdots \qquad I_{t_2} = \int_0^{t_2} \cdots$

$$\mathcal{V}(0,t_1) \hookrightarrow \mathcal{V}(0,t_2)$$

$V \longmapsto \bar{V}$ where $\bar{V}(t) = V(t),\ t \le t_1$

$\bar{V}(t) = 0,\ t \ge t_1$

$$I_{t_2}(\bar{V},\bar{V}) = I_{t_1}(V,V) \Rightarrow i(t_2) \geq i(t_1)$$

Since we can move the t_i a little bit, we can assume, for fixed t, that

$$\exists i \qquad t \in (t_{i-1}, t_i)$$

<u>Lemma 1:</u> If $\varepsilon > 0$ is small enough, then

$$i(t-\varepsilon) = i(t)$$

<u>Lemma 2:</u> If $\varepsilon > 0$ is small enough, then

$$i(t+\varepsilon) = i(t) + n(t)$$