$$I_{t_2}(\bar{V},\bar{V}) = I_{t_1}(V,V) \Rightarrow i(t_2) \geq i(t_1)$$

Since we can move the t_i a little bit, we can assume, for fixed t, that

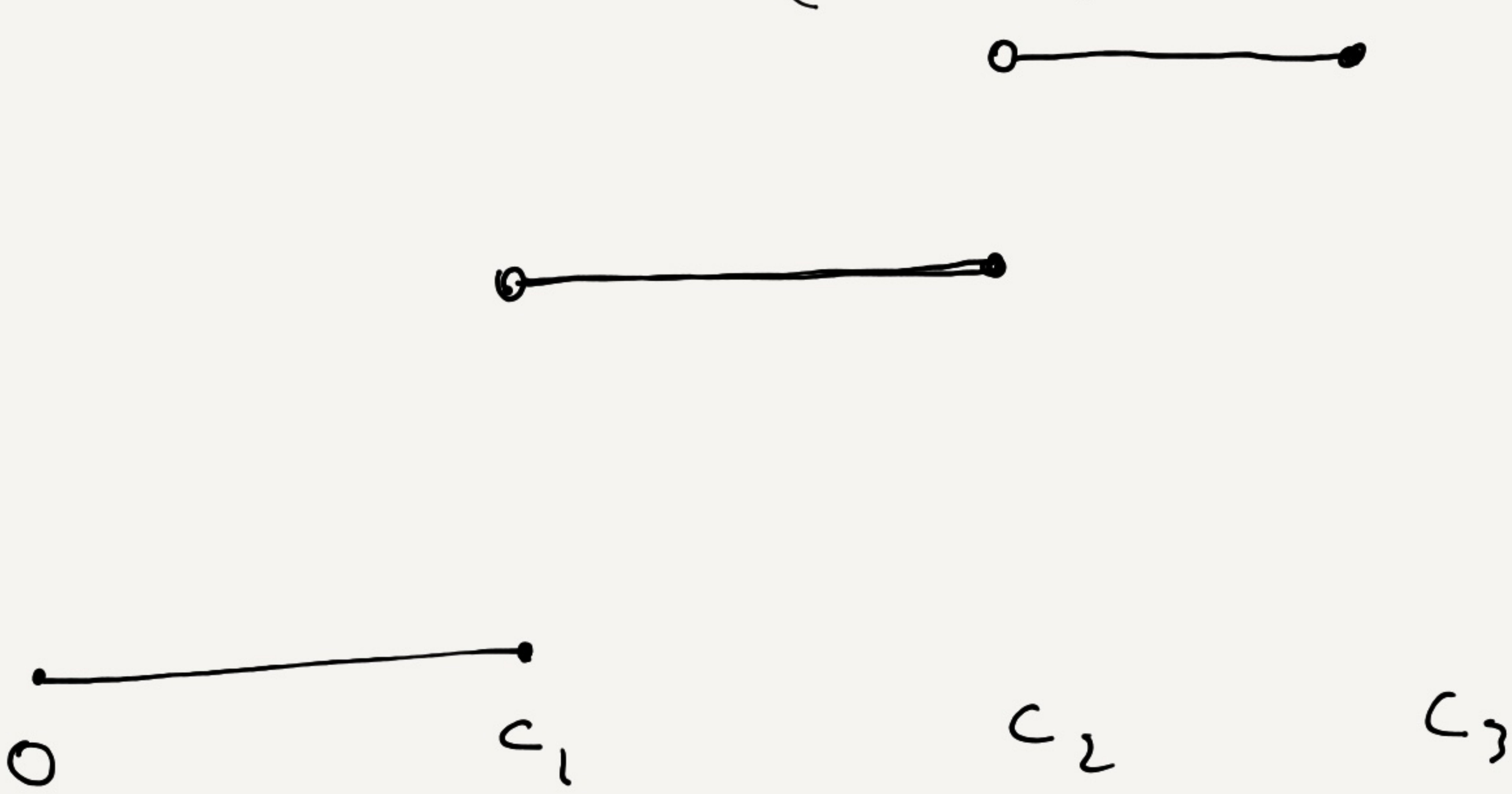
$\exists i \quad t \in (t_{i-1}, t_i)$

<u>Lemma 1:</u> If $\varepsilon > 0$ is small enough, then

$$i(t-\varepsilon) = i(t)$$

<u>Lemma 2:</u> If $\varepsilon > 0$ is small enough, then

$$i(t+\varepsilon) = i(t) + n(t)$$



and $n(t)$ = multiplicity of t as a conjugate point.

<u>Proof of Lemma 1:</u> $t \in (t_{i-1}, t_i)$, then

$$V^-(0,t) \simeq T_{\gamma(t_1)}M \oplus \dots \oplus T_{\gamma(t_{i-1})}M$$

Recall the formula:

$$I_t(V,W) = \int_0^t -\left\langle \frac{D^2V}{dt^2} + R(\gamma',V)\gamma', W\right\rangle dt - \sum_{j=1}^{i-1} \left\langle \frac{DV}{dt}(t_j^+) - \frac{DV}{dt}(t_j^-), W(t_j)\right\rangle$$

I_t is a bilinear form on

$$\mathcal{V}^-(0,t) \cong \underbrace{T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_{i-1})}M}_{S_i \text{ fixed for } t\in(t_{i-1},t_i)}$$

Since the elements of $\mathcal{V}^-$ are Jacobi fields, we have on $\mathcal{V}^-$:

$$I_t(V,W) = -\sum_{j=1}^{i-1} \left\langle \frac{DV}{dt}(t_j^+) - \frac{DV}{dt}(t_j^-), W(t_j)\right\rangle$$

As we move t between t_{i-1} and t_i, the Jacobi fields on $[t_{i-1}, t]$ move as do the values $V(t_{i-1})$, $\frac{DV}{dt}(t_{i-1}^+)$: they move continuously with t.

So I_t moves continuously with t.

So if I_t is negative definite on some subspace $S \subset S_i$, then $\exists\, \eta$ s.t. $\forall\, \varepsilon < \eta$, $\varepsilon > 0$, $I_{t \pm \varepsilon}$ is neg. def. on $S \Rightarrow i(t-\varepsilon) \geq i(t)$

i is a non decreasing function

$\Rightarrow \forall\, \varepsilon < \eta$, $i(t-\varepsilon) = i(t)$ $\square$

Proof of Lemma 2:

I_t is a bilinear form on S_i which has dim. $n(i-1)$, so I_t is positive definite on a subspace S of dim. $n(i-1) - i(t) - n(t)$

$\exists\, \eta > 0$ s.t. $\forall\, \varepsilon > 0$, $\varepsilon < \eta$

$I_{t \pm \varepsilon}$ is positive definite on S

$$\Rightarrow i(t+\varepsilon) \leq n(i-1) - \big(n(i-1) - i(t) - n(t)\big)$$
$$= i(t) + n(t)$$

Now we prove $i(t+\varepsilon) \geq i(t) + n(t)$

Note that $i(t) + n(t)$ is the maximum

dimension of a subspace S of S_i on which I_t is negative, i.e., $\forall\, V \in S$

$$I_t(V,V) \leq 0.$$

Choose such an S and $V \in S$.

Under the identification

$$\mathcal{V}^-(0,t) = S_i = \mathcal{V}^-(0,t+\varepsilon),$$

let W be the vector field on $[0,t+\varepsilon]$ corresponding to W. In other words, V and W take the same value at all the t_j, $j=1,\dots,i-1$.

Let $\bar{V}$ be the extension of V to $[0,t+\varepsilon]$ by 0. Then

$$I_t(V,V) = I_{t+\varepsilon}(\bar{V},\bar{V})$$

and, by the Index Lemma of Chapter 10:

$$I_{t+\varepsilon}(\bar{V},\bar{V}) > I_{t+\varepsilon}(W,W)$$

because W is a Jacobi field on $[t_{i-1}, t+\varepsilon]$ and $\bar{V}$ is not.

Hence $I_t(V,V) > I_{t+\varepsilon}(W,W)$,

$$\Rightarrow\ 0 > I_{t+\varepsilon}(W,W)$$

and $i(t+\varepsilon) \geqslant i(t) + n(t)$

$\square$

The Morse index theorem now follows:

$\forall\ t$, $n(t) = \#$ Jacobi fields on $[0,t]$ that vanish at t

$= $ multiplicity of $\gamma(t)$ as a conjugate point of $\gamma(0)$

Hence, if $s_1, \dots, s_m$ are the coordinates of the points of γ conjugate to $\gamma(0)$,

$$
\begin{aligned}
i(I) = i(a) &= i(s_m) + n(s_m) \\
&= i(s_{m-1}) + n(s_{m-1}) + n(s_m) \\
&\ \vdots \\
&= n(s_1) + \cdots + n(s_m)
\end{aligned}
$$

$\square$