ON THE MODULI SPACE OF FOUR-DIMENSIONAL PRINCIPALLY POLARIZED ABELIAN VARIETIES

by

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ABSTRACT

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties (ppav) of dimension $g$ over the field $\mathbb{C}$ of the complex numbers. Our aim is to study the structure of $\mathcal{A}_4$ and some of its relations with $\mathcal{A}_5$.

Let $\mathcal{J}_{\text{hyp}}$ be the locus of jacobians of hyperelliptic curves in $\mathcal{A}_4$ and let $\overline{\mathcal{J}_{\text{hyp}}}$ be its closure in $\mathcal{A}_4$. Our main tool will be a cubic threefold $T$ with an "even" double cover $1 : \tilde{\mathcal{F}} = \mathbb{P}^{-1}(A) \to \mathcal{F}$ of the Fano variety of lines $\mathcal{F}$ of $T$ which we will associate to each element $A$ of $\mathcal{A}_4 \setminus (\overline{\mathcal{J}_{\text{hyp}}} \cup \mathcal{A}_{n11})$ (see the text for the definition of $\mathcal{A}_{n11}$). The pair $(T,\mu)$ (where $\mu$ is the point of order 2 associated to 1) was first introduced by Donagi for generic ppav’s in an abstract way. He used points of order 2 on plane quintics.

Using a construction of Clemens for double solids we first define $(T,l)$ as a hypersurface in $\mathcal{H}_{10}$ for a generic ppav $A$. Here $\Theta$ is a symmetric theta-divisor on $A$ and $\mathcal{H}_{10}$ is the sublinear system of $|2\Theta|$ consisting of those divisors which have a point of multiplicity greater than or equal to 4 at 0. Then we extend the definition of $(T,l)$ to $\mathcal{A}_4 \setminus (\overline{\mathcal{J}_{\text{hyp}}} \cup \mathcal{A}_{n11})$.

Let $h : A \to (|2\Theta|_{00})^*$ be the natural map. Let $\tilde{\mathcal{A}}$ be the blow up of $A$ at 0 and let $\tilde{h}$ be the lift of $h$ to $\tilde{\mathcal{A}}$. From theorem 4 below it follows that $\tilde{h}$ is a morphism everywhere on $\mathcal{A}_4 \setminus (\overline{\mathcal{J}_{\text{hyp}}} \cup \mathcal{A}_{n11})$. Let $E$ be the exceptional divisor in $\tilde{\mathcal{A}}$. For a generic ppav we show:

**Theorem 1**: The branch locus of $\tilde{h}$ is the union of $\tilde{h}(E)$ and the dual variety $T^*$ of $T$.

There is a three-dimensional family of double solids (with six ordinary double points in general position) with intermediate jacobian a given ppav $A$. The first use of the cubic threefold will be
Theorem 2: One can recover all the double solids with intermediate jacobian $A$ from the data of $A$.

Let $\tau|2\Theta|_{00}$ be the linear system of quartic tangent cones to elements of $|2\Theta|_{00}$. In relation with the Schottky problem of characterizing jacobians of curves in $\mathcal{C}_4$ we prove:

Theorem 3: Let $A \in \mathcal{C}_4 \setminus (\overline{\mathcal{J}_4} \cup \mathcal{C}_{n11})$, then

(i) The only base point of $|2\Theta|_{00}$ is $0$ with multiplicity $4^4$.

(ii) The base locus of $\tau|2\Theta|_{00}$ is empty. In particular, $\tau|2\Theta|_{00}$ has always dimension $\geq 3$.

We also determine the loci on which $T$ is singular. Let $\theta_{\text{null}}$ be the locus of ppav's with a vanishing theta-null.

Theorem 4: Let $A \in \mathcal{C}_4 \setminus (\overline{\mathcal{J}_{\text{hyp}}} \cup \mathcal{C}_{n11})$. The cubic threefold $T$ associated to $A$ is singular if and only if $A \in \theta_{\text{null}} \cup \mathcal{J}_4$.

From this we deduce

Corollary: There is a finite rational map of degree $2^4(2^3+1) - 1 + 2^4(2^3+1) - 1$

$$\theta_{\text{null}} \rightarrow \mathcal{J}_4$$

Using the cubic threefold we then complete results of Clemens and Donagi by determining the double solids above a generic element of $\theta_{\text{null}} \cup \mathcal{J}_4$.

At the end, we gather a few more results that could be useful for further developments.
To my family
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1. INTRODUCTION

Let \( \mathcal{A}_g \) be the moduli space of principally polarized abelian varieties (ppav) of dimension \( g \) over the field \( \mathbb{C} \) of the complex numbers. Our aim is to study the structure of \( \mathcal{A}_4 \) and some of its relations with \( \mathcal{A}_5 \). We summarize some of our results in a diagram, which we now describe.

For a family of ppav's parametrized by a variety \( S \) we denote by \( S^0 \) the open subset of \( S \) parametrizing "automorphism free" abelian varieties, i.e., those whose only automorphisms are translations and \( \pm \)identity.

Let \( \mathcal{K} \) be the universal Kummer variety over \( \mathcal{A}_4^0 \). For an element \( A \) of \( \mathcal{A}_4 \) with symmetric theta divisor \( \Theta \) we denote by \( \Gamma \) the vector space \( H^0(A, \mathcal{O}(2\Theta)) \), by \( \Gamma_{00} \) the space of sections of \( \Gamma \) vanishing at 0 with multiplicity greater than or equal to 4 and by \( |2\Theta|_{00} \) the projectivisation of \( \Gamma_{00} \). For an indecomposable abelian variety, i.e., one that is not the product of two ppav's of lower dimensions, the dimension of \( \Gamma_{00} \) is 5 (see [I] p. 188). An automorphism free ppav is a fortiori indecomposable.

Let \( \mathcal{P} \) be the bundle over \( \mathcal{A}_4^0 \) with fiber \( |2\Theta|_{00} \) at \( A \); likewise \( \mathcal{P}^* \) is the bundle with fibers \( (|2\Theta|_{00})^* \). So we have a commutative diagram of canonical maps:

\[
\begin{array}{ccc}
\mathcal{K} & \to & \mathcal{P}^* \\
\downarrow & & \downarrow \\
\mathcal{A}_4^0 & & \\
\end{array}
\]

Let \( \mathcal{P}_{g+1} \) be the moduli space of curves \( X \) of genus \( g+1 \) with a distinguished point \( \eta \) of order 2 in the jacobian \( JX \) of \( X \). For \( (X, \eta) \) in \( \mathcal{P}_{g+1} \), if \( \bar{X} \) is the double cover of \( X \) determined by \( \eta \), then the covering involution \( \sigma : \bar{X} \to \bar{X} \) induces
an involution (still denoted by $\sigma$) on $\mathcal{J}X$. The Prym variety $P(X, \eta)$ or $P(\mathcal{J}X, X)$ is the image of $\sigma - \text{id}: \mathcal{J}X \to \mathcal{J}X$. The Prym variety is a ppav of dimension $g$. The map which to each $(X, \eta)$ associates its Prym variety is called the Prym map and denoted by $P$. If for an abelian variety $B$ we denote by $B_2$ its group of points of order 2, we have an exact sequence ([M3]):

$$0 \to \{\eta\} \to \{\eta\}^\perp \to P(X, \eta)_2 \to 0$$

where $^\perp$ means orthogonal complement with respect to the symmetric and antisymmetric bilinear form on $(\mathcal{J}X)_2$ (see [I] p. 214 and [M2]).

The map $P$ is generically surjective [B1] so as $\mathcal{P}_5$ is of dimension 12 and $\mathcal{Q}_4$ of dimension 10, the fibers of $P: \mathcal{P}_5 \to \mathcal{Q}_4$ are generically of dimension 2.

We call $\Sigma(X, \eta)$ (or $\Sigma(X)$ when $P(X, \eta)$ is fixed and there is no ambiguity on $\eta$) the surface image of the symmetric product $\tilde{X}^{(2)}$ by the map

$$\tilde{X}^{(2)} \to A$$

$$(p, q) \mapsto [p, q] = p + q - \sigma p - \sigma q$$

(or for short $\Sigma(X)$ when $P(X, \eta)$ is fixed and there is no ambiguity on $\eta$). We show that the morphism

$$\Sigma_A = \bigcup_{(X, \eta) \in P^{-1}A} \Sigma(X, \eta) \to A$$

is of degree 27 (outside the origin). Actually, if $\tilde{A}$ is the blow up of $A$ at the origin, this morphism lifts to a morphism $\bigcup \tilde{X}^{(2)} \to \tilde{A}$.

So we have an induced morphism of degree 54 (outside the origin) from $\Sigma_A$ to $K(A) = A/\pm \text{id}$ and a commutative diagram:

\[
\begin{array}{ccc}
(1) & \cdots & \Xi = \bigcup_A \Sigma_A & \cdots & (3) \\
\downarrow & & \downarrow & & \\
\mathcal{K} & \mathcal{P}_5 & \mathcal{Q}_4 & \mathcal{P}^* \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
(2) & \cdots & \cdots & \cdots & \cdots
\end{array}
\]
We are going to fill in the "empty corners" of the above.

1.1 - The corner (1) of the diagram

For a generic curve $X$ of genus 5 there is a net of quadrics containing its canonical model $\kappa X$ and a smooth plane quintic $Q$ parametrizes the singular quadrics in the net. To each bisection $\langle p, q \rangle$ of $\kappa X$ we associate the pencil 1 of quadrics containing $\langle p, q \rangle$ and $\kappa X$.

So if we let $\tilde{\mathcal{P}}_5$ be the moduli space of triples $(X, \eta, l)$ where $(X, \eta) \in \mathcal{P}_5$ and $l$ is a line in the net of quadrics containing $\kappa X$, there is a map $\Xi \to \tilde{\mathcal{P}}_5$ of degree 64 and we put $\tilde{\mathcal{P}}_5$ in (1) with its canonical projection onto $\mathcal{P}_5$.

1.2 - The corner (2) of the diagram

Consider an abelian variety $A \in \mathcal{A}_4$, recall that $\tilde{A}$ is its blow up at 0. We show that if $A$ is neither in the closure $\overline{\mathcal{J}}_4$ of the locus $\mathcal{J}_4$ of jacobians of curves nor in a certain subvariety $\mathcal{A}_n^{11}$ of $\mathcal{A}_4$ (see section 7), then the $\Gamma_{00}$-map extends to a morphism $\tilde{h}$ from $\tilde{A}$ to $(|2\Theta|_{00})^*$ of generic degree $2^7$. The branch locus of $\tilde{h}$ has 2 components:

- the image $R_0$ of the exceptional divisor
- the dual of a certain cubic threefold $T$ in $|2\Theta|_{00}$ arising in a natural way:

Consider the moduli space $\mathcal{Z}$ of quartic double solids with six ordinary nodes. A double solid $Z$ is a double cover of $\mathbb{P}^3$ branched along an even degree surface. A quartic double solid has as branch locus a quartic surface. When the quartic surface is smooth, the intermediate jacobian

$$JZ = H^{2,1}(Z)^*/H_5(Z)$$

is a ppav of dimension 10. When the branch locus acquires ordinary double points in general position the rank of $JZ$ drops 1 for each double point. So for a quartic double
solid with 6 ordinary double points in general position, \( JZ \) is an element of \( \mathcal{Q}_4 \).

The rational map \( J : \mathcal{Z} \to \mathcal{Q}_4 \) is generically surjective [C1] hence its generic fibers have dimension 3.

To each \( Z \in J^{-1}(A) \) Clemens associates an element \( D_Z \) of \( |2\Theta|_{00} \) (see 2.1). We have a commutative diagram

\[
\begin{array}{c}
D \\
\mathcal{Z} \to \mathbb{P} \\
\downarrow J \quad \downarrow \\
\mathcal{Q}_4^0
\end{array}
\]

We show that the image \( D(J^{-1}(A)) \) is the cubic threefold \( T \). The cubic threefold \( T \) comes with a point \( \mu \) of order 2 in its intermediate jacobian.

Let \( p_1, \ldots, p_6 \) be the double points of a double solid \( Z \). Let \( \tilde{Z} \) be the blow up of \( Z \) at its double points. The threefold \( \tilde{Z} \) admits 42 conic bundle structures over \( \mathbb{P}^2 \) which permit us to write \( JZ = P(X, \eta) \) in 42 distinct ways. For 12 of these, say \((X_i, \eta_i)\), \((X_i', \eta_i')\) for \(i \in \{1, \ldots, 6\}\), the double covers \( \tilde{X}_i \) and \( \tilde{X}_i' \) parametrize respectively lines through \( p_i \) and twisted cubics through \( p_j \) (for all \( j \neq i \)) in \( Z \).

We show that there are embeddings of \( \Sigma(X_i, \eta_i) \) and \( \Sigma(X_i', \eta_i') \) in \( D_Z \).

Fixing \((X, \eta)\) there is a one dimensional family of double solids \( Z \) such that \((X, \eta) = (X_i, \eta_i) \) for some \( i \), then \((X_i', \eta_i')\) does not depend on the double solid \( Z \) in this family and is denoted by \((X_\lambda, \eta_\lambda)\). By an argument of homology class we deduce that the one dimensional family of double solids gives us a pencil \( l_\lambda \) of \( \Gamma_{00} \)-divisors \( D_Z \) in \( T \) containing \( \Sigma(X, \eta) \) and \( \Sigma(X_\lambda, \eta_\lambda) \).

Hence we obtain a 2-to-1 map \( 1: P^{-1}(A) \to F \), \( F \) being the Fano surface of lines of \( T \). The involution acting in the fibers of this map is \( \lambda \). This map and the involution \( \lambda \) were first introduced by Donagi in a different way.

The double cover \( P^{-1}(A) \to F \) defines a point \( \mu \) of order 2 in the albanese
variety of \( F \) which is isomorphic to \( JT ([CG]) \). We will see below that \( \mu \) is even with respect to the standard quadratic form on \( (JT)_2 \), where for an abelian variety \( B, B_2 \) denotes the set of points of order 2.

This gives a birational correspondence (first defined abstractly by Donagi) between \( \mathcal{C}_4 \) and the moduli space \( \mathcal{G}_2 \) of cubic threefolds with an even point of order 2 in their intermediate jacobian.

Letting \( \mathbb{P} \to \mathbb{P}^* \) be defined by the partial derivatives of the cubic threefolds \( T \) in each fiber and \( T_2 \) be the universal cubic threefold over \( \mathcal{G}_2 \), we can fill the corner (2) of our diagram:

\[
\begin{array}{c}
\mathcal{C}_4 \downarrow \\
\mathcal{T}_2 \end{array}
\]

\[
\begin{array}{c}
\mathcal{C}_4 \leftarrow \mathcal{G}_2
\end{array}
\]

\[
\begin{array}{c}
\mathbb{P}^* \leftarrow \mathbb{P}^* \leftarrow \mathbb{P}
\end{array}
\]

\[
\begin{array}{c}
\mathbb{P}^* \leftarrow \mathbb{P}
\end{array}
\]

1.3 - The corner (3) of the diagram

We give an interpretation of the \( \Gamma_{00} \)-map \( \tilde{h} \) in terms of Prym-embeddings of curves in intersections of two translates of \( \Theta \).

Let \( (\tilde{X}, X) = (X, \eta) \) be in \( \mathbb{P}^1(A) \). The antisymmetric part of \( \tilde{J}X \) for the involution \( \sigma \) induced by the covering involution on \( \tilde{X} \) is the union of the "even part" \( A = P(X, \eta) \) and "the odd part" \( A^- \), a translate of \( A \) by an odd point of order 2 in \( \tilde{J}X \) ([M2]). (Odd here means odd with respect to the standard \( \mathbb{Z}/2\mathbb{Z} \)-valued quadratic form on the points of order 2 in \( \tilde{J}X \) (see [I] p. 214 and [M2]).)

The set \( \{ p - \sigma p ; p \in \tilde{X} \} \) is contained in \( A^- \). A Prym-embedding of \( \tilde{X} \) in \( A \) is by definition a translate of \( \{ p - \sigma p ; p \in \tilde{X} \} \) by an element of \( A^- \).

For an element \( x \) of \( A \), let \( t_x \) be translation by \( x \) in \( A \), i.e., \( t_x(y) = x + y \) for
all \( y \in A \). Let \( \Theta_x = (t, x)^* \Theta \) be the translate of \( \Theta \) by \( x \). We show that for a generic element \( x \) of \( A \) the intersection \( \Theta \Theta_x \) contains exactly two Prym-embeddings of 27 distinct Prym curves for \( A \) and that the lines in \( T \) corresponding to these curves are the lines in the hyperplane section of \( T \) given by \( \tilde{h}(x) \). This is equivalent to the fact that \( \Sigma_A \to A \) has degree 27.

At this point we need to introduce Donagi's tetragonal construction ([Do1]):

For a curve \( C \) we denote by \( g_d^r \) the linear systems of degree \( d \) and projective dimension \( r \) on \( C \). Consider a curve \( C \) of genus \( g \) with a \( g_4^1 \) and an etale double cover \( \tilde{C} \). The curve \( \tilde{C} \) is of genus \( 2g-1 \). The set of liftings of divisors of \( g_4^1 \) in \( \tilde{C}^{(4)} \) is a curve. This curve splits into two irreducible components \( \tilde{C}' \) and \( \tilde{C}'' \): intuitively \( \tilde{C}' \) parametrizes the liftings which have an even number of points from either sheet of the cover \( \tilde{C} \to C \), and \( \tilde{C}'' \) parametrizes the liftings which have an odd number of points from either sheet of the cover. The curves \( \tilde{C}' \) and \( \tilde{C}'' \) each come with an involution say \( \sigma' \) and \( \sigma'' \). These involutions interchange complementary liftings of the same divisor in \( g_4^1 \). The quotients \( C' = \tilde{C}' / \sigma' \) and \( C'' = \tilde{C}'' / \sigma'' \) have genus \( g \), and both \( C' \) and \( C'' \) have \( g_4^1 \)'s on \( C' \) for instance the points of a divisor of the \( g_4^1 \) are the classes modulo \( \sigma' \) of the "even" liftings of a given divisor of the \( g_4^1 \) on \( C \).

Moreover the construction is symmetric.

The tetragonal construction of Donagi [Do1] and the trigonal construction of Recillas [R] are interesting special cases of a more general construction in [B3].

For the double solid \( Z \) the curves \( (X_i, \eta_i) \) and \( (X_j, \eta_j) \) are tetragonally related whenever \( i \neq j \). For fixed \( i \) and \( j \) the third curve \( (X_{ij}, \eta_{ij}) \) in the tetragonal relation is a discriminant curve for one of the other 30 conic bundle structures of \( \tilde{Z} \). More precisely \( \tilde{X}_{ij} \) parametrizes pairs of incident lines in \( \tilde{Z} \), one of which is an element of \( \tilde{X}_i \) and the other an element of \( \tilde{X}_j \). So 15 curves occur in this way.

Similarly \( (X_i, \eta_i) \) and \( (X_j, \eta'_j) \) are tetragonally related whenever \( i \neq j \). The third
curves \((X_{ij}^i, n_{ij})\) in the tetragonal relations give the other 15 conic bundle structures on \(\tilde{Z}\).

We have \((X_{ij}^i, n_{ij}^i) = \lambda (X_{ij}, n_{ij})\). The 2 \(g_4^1\)'s on \(X\) relating \((X_i, n_i)\) to \((X_j, n_j)\) and \((X_j^i, n_j^i)\) are opposite, that is if we denote them by \(g\) and \(h\) then \(g + h \equiv K_{X_i}\) where \(K_{X_i}\) is a canonical divisor on \(X_i\) ("\(\equiv\" denotes linear equivalence of divisors).

Fix an element \((X, \eta)\) of \(\mathbb{P}^1(A)\). We denote by \(\text{Pic}^dX\) the principal homogeneous space on \(JX\) parametrizing linear systems of degree \(d\) on \(X\). By a suitable isomorphism \(JX \equiv \text{Pic}^dX\), each \(g_4^1\) on \(X\) goes to a singular point of the theta divisor \(\Theta'\) of \(JX\). The tangent cone to the theta divisor at a \(g_4^1\) is a singular quadric containing \(\kappa X\). This defines a 2 to 1 map from the singular locus \(\text{Sing}\Theta'\) of \(\Theta'\) onto the plane quintic \(Q\). Under this map each \(g_4^1\) and its opposite \(K_X - g_4^1\) go to the same singular quadric \(Q\). A given \(Q\) is of rank less than or equal to 4, if the rank is exactly 4 it has two rulings by planes. One ruling cuts on \(\kappa X\) the divisors of \(g_4^1\) and the other cuts on \(\kappa X\) the divisors of \(K_X - g_4^1\) [AM]. Also \(JX\) is the Prym variety of the double cover \(\text{Sing}\Theta' \to Q\).

Back to the threefold \(T\) and its family of lines, we reprove the fact that two Prym curves are tetragonally related if and only if their lines are incident. Originally Donagi proved this on the threefold \(T\) which he associated to \(A\) by an intricate indirect construction. Let \(l_X\) be the line in \(T\) associated to \(X\). The projection from \(l_X\)

\[\frac{2\Theta}{\Theta} \to \mathbb{P}^2\]

defines a conic bundle structure on \(T\) with discriminant curve \(Q\). For a \(\Gamma_0\)-divisor \(D\) let \(q\) be the quadric in \(\lfloor K_X \rfloor^*\) that is its image by the projection from \(l_X\). We show that for each \(D\), \(D \cap \Sigma(X)\) is the set of points that give bisecant lines to \(\kappa X\) that are contained in \(q\).

The family of lines incident to \(l_X\) is a double cover \(\tilde{Q}\) of \(Q\) with Prym variety
the intermediate jacobian of $T$. The two points of $\overline{Q}$ over a point $q$ of $Q$ correspond to the two lines in $T$ in the plane section of $T$ corresponding to $q$. The point $\alpha$ of order 2 on $JQ$ corresponding to this double cover is odd.

From this we conclude that $(X,\eta)$ and $(X_\lambda,\eta_\lambda)$ have the same plane quintic $Q$. We obtain three points of order 2 in $JQ$ for the three double covers of $Q$ with Pryms $JT$, $JX$ and $JX_\lambda$. We show that these points are the three elements of a rank 2 vector space on $F_2 = \mathbb{Z}/2\mathbb{Z}$ contained in $(JQ)_2$. This subspace is totally isotropic with respect to the symmetric and antisymmetric bilinear form on $(JQ)_2$. (This is Donagi's starting point)

Let $Q$ be the moduli space of triples $(Q,V^2,1)$ where $Q$ is a plane quintic, $V^2$ is a rank 2 vector space on $F_2$ contained in $(JQ)_2$ such that $V^2$ is totally isotropic with respect to the symmetric and antisymmetric bilinear form on $(JQ)_2$ and $V^2$ contains an odd point and two even points, 1 is a line in the plane of $Q$.

We define a map from $\Xi$ to $Q$ in the following natural way:

Let $[p,q] \in \Sigma(X,\eta)$ ($\langle X,\eta \rangle \in P^{-1}(A)$) then the image of $[p,q]$ is the triple $(Q,V^2,1)$ where $Q$ is the quintic parametrizing singular quadrics containing $\kappa X$, $V^2$ is the inverse image of $\eta$ by the last map in the exact sequence associated to the Prym construction $P(Sing\Theta',Q) = JX$

$$0 \rightarrow \{\alpha'\} \rightarrow \{\alpha'\perp} \rightarrow (JX)_2 \rightarrow 0$$

and 1 is the pencil of quadrics containing $\kappa X$ and the line $\langle p, q \rangle$ in the canonical space of $X$.

Given $(Q,V^2,1)$ we get the abelian variety $A$ from $(Q,V^2)$ by two successive Prym constructions, and a hyperplane section of $|2\Theta|_{00}$ from 1. Hence a map from $Q$ to $\mathbb{P}^*$ of degree 27. We can now complete our diagram:
1.4 - Relation of the diagram with $\mathcal{Q}_5$ and $\mathcal{M}_6$

We let $\mathcal{M}_g$ be the moduli space of curves of genus $g$. For all families of ppav's of dimension $g$ parametrized by a variety $M$ we denote by $\overline{M}$ the partial compactification of $M$ by $C^*$-extensions of $(g-1)$-dimensional ppav's with properties similar to those of elements of $M$. We denote by $M^0$ (resp. $\overline{M}^0$) the open subset of "automorphism free" group schemes.

Then $\mathcal{K} \subseteq \overline{\mathcal{Q}}_5^0$ is the boundary:

$$\mathcal{K} = \overline{\mathcal{Q}}_5^0 \setminus \mathcal{Q}_5^0$$

because if we let $G_m$ be the multiplicative group then we have ([M1], page 227):

$$A \cong \text{Pic}^0 A \cong \text{Ext}^1(A, G_m)$$

where $\text{Pic}^0 A$ is the dual abelian variety of $A$ or the abelian variety parametrizing homologically trivial line bundles on $A$ and $\text{Ext}^1(A, G_m)$ is the group of $(G_m(C) = C^*)$-extensions of $A$. Two extensions of $A$ by $C^*$ with extension data's $b, b' \in A$ are isomorphic if and only if there exists an automorphism $\nu$ of $A$ which is not a translation such that $\nu(b) = b'$. So for $A \in \mathcal{Q}_4^0$ the isomorphism classes of $C^*$-extensions of $A$ are parametrized by $K(A) = A/\text{Hd}$.

Let $\mathcal{Q}_{g,2}$ denote the moduli space of ppav's of dimension $g$ with a nontrivial point of order 2. Let $\overline{\mathcal{Q}}_{5,2}$ (resp. $\overline{\mathcal{Q}}_5$) be the blow up of $\overline{\mathcal{Q}}_{5,2}$ (resp. $\overline{\mathcal{Q}}_5$) along the loci $\mathcal{P}_5$ (resp. $\mathcal{J}_5$ of jacobians) and $\mathcal{S}_2$ (resp. $\mathcal{T}$) of intermediate jacobians of cubic threefolds.
We can relate all these maps and spaces to our diagram (*) as follows:

1) \( \tilde{\mathcal{F}}_5 \subset \tilde{\mathcal{Q}}_{5,2} \) is the strict transform of \( \mathcal{P}_5 \). This is because the conormal space to the jacobian locus at a jacobian \( JX \) can be canonically identified with \( I \) (the vector space of quadrics containing \( \kappa X \)) by the exact sequence

\[
0 \to I \to S^2H^0(X,\omega_X) \to H^0(X,(\omega_X)^2) \to 0
\]

(here \( \omega_X \) is the canonical sheaf of \( X \)).

2) \( \mathcal{P}^* \subset \tilde{\mathcal{Q}}_{5,2} \) is the strict transform of the locus \( S_2^2 \subset S_2 \) of intermediate jacobians of cubic threefolds with an even point of order 2. (This is because of the fact that the conormal space of \( S_2 \) in \( \tilde{\mathcal{Q}}_5 \) at a cubic threefold \( T \) can be canonically identified with the ambient \( \mathcal{P}^4 \) of \( T \) [DS].)

3) If \( \mathcal{P}_6^2 \) is the moduli space of curves \( C \) of genus 6 with a totally isotropic subspace of rank 2 in \( (JC)_2 \) (with two even points of order 2 and one odd point of order 2) and we let \( \tilde{\mathcal{P}}_6^2 \) be the blow up of \( \mathcal{P}_6^2 \) along the locus of plane quintics, then we have \( \mathcal{Q}_6 \subset \tilde{\mathcal{P}}_6^2 \). This follows from the fact that the conormal space to the locus of plane quintics in \( \mathfrak{M}_6 \) at a point \( Q \) can be canonically identified with the ambient \( \mathcal{P}^2 \) of \( Q \) [DS].

4) The map \( \mathcal{Q}_6 \to \mathcal{P}^* \) in our diagram is induced by the Prym map.

5) Let \( \tilde{\mathcal{P}}_5 \) project onto \( \tilde{\mathfrak{M}}_5 \cong \tilde{\mathcal{Q}}_5 \) in \( \tilde{\mathcal{Q}}_5 \) and \( \mathcal{P}^* \) project onto \( (\mathcal{P}')^* \), then we have a diagram of correspondences between subvarieties of \( \tilde{\mathcal{Q}}_5 \), all dominated by \( \Xi \):

\[
\Xi \\
\vee \downarrow \nearrow \\
\tilde{\mathfrak{M}}_5 \mathcal{K} \to (\mathcal{P}')^*
\]

1.5 - Prym curves over the boundary of \( \mathcal{Q}_5 \) and

\[ \Theta_x, \Theta_x \in A \in \mathcal{Q}_4 \]

(1.5.1) By [DS] the degree of the Prym map (still denoted by \( P \)) from \( \mathcal{P}_5 \) to \( \mathcal{Q}_5 \) is 27.
Recall that $\mathcal{K} = \tilde{\mathfrak{A}}_5 \setminus \mathfrak{A}_5$ is the boundary. The inverse image of $\mathcal{K}$ under the extension of $P$ to the moduli space of stable curves with a point of order 2 is the variety $\text{Gen}P_6$ of generalized jacobians of singular irreducible curves with one node and a double cover with two nodes exchanged by the covering involution.

For each Prym-curve $\tilde{X}$ in $\Theta, \Theta_x$ write $x = [p,q] \in \Sigma(X,\eta)$, then the extension $A_x$ of $A$ corresponding to $x$ is the Prym variety of the double cover $\tilde{X}_{pq} = \tilde{X}/\sigma = \sigma_q, \sigma_p = q$ of the curve $X_{pq} = X/\pi_p = \pi_q$. We show:

The extension of $P$ to $\text{Gen}P_6 \cup P_6$ is generically unramified on $\text{Gen}P_6$.

So using the properness of $P$ [B1], we obtain a second proof of the result of [DS] by combining the theorem with the fact that a generic intersection of two translates of $\Theta$ in $A$ contains 27 Prym curves.

(1.5.2) Let $B$ be an element of $\tilde{\mathfrak{A}}_5$. The tetragonal relation defines a correspondence between the 27 elements of $P(B)$ which is similar to the incidence correspondence among the lines in a cubic surface in $\mathbb{P}^3$. A natural question is:

Can one produce a family of cubic surfaces on $\tilde{\mathfrak{A}}_5$ and an $\tilde{\mathfrak{A}}_5$-isomorphism between $\tilde{\mathfrak{P}}_5$ and the family of lines in these surfaces?

Such a family is easily produced generically on the strict transform $(\mathbb{P}')^* \cup (\mathbb{P}'')^*$ of $\mathfrak{N}$: an element $B \in (\mathbb{P}')^* \cup (\mathbb{P}'')^*$ consists of a pair $(T,n)$ where $T \in \mathfrak{N}$ and $n$ is a normal direction to $\mathfrak{N}$ in $\mathfrak{A}_5$ hence $n$ defines a hyperplane section of $T$ which is the desired cubic surface.

We can produce this family on $\mathcal{K}$: an element $B$ of $\mathcal{K}$ is an extension of a four-dimensional ppav $A$ with extension data $b \in A$. Then $\tilde{h}(b) \in (2\Theta_{b_0})^*$ gives a hyperplane section of the cubic threefold $T$ associated to $A$ which is the cubic surface we are looking for.

A family of cubic surfaces cannot be produced on $\tilde{\mathfrak{F}}_5$ but instead on $\tilde{\mathfrak{P}}_5$: let $(X,\eta,l) \in \tilde{\mathfrak{P}}_5$, then by inverse image in the exact sequence
\[ 0 \rightarrow \{\alpha'\} \rightarrow \{\alpha'\}^\perp \rightarrow (JX)_2 \rightarrow 0 \]

associated to the Prym construction \( P(\text{Sing}\Theta', Q) = P(Q, \alpha') = JX \) one gets as before a rank 2 totally isotropic subspace of \((JQ)_2\) and an odd point \(\alpha\) of order 2 in this subspace (see 5.29). Hence \( P(Q, \alpha) = JT \) for a cubic threefold \( T \). We also have a projection

\[
\begin{array}{c}
\text{T} \subset \mathbb{P}^4 \\
\downarrow \\
\mathbb{P}^2 \supset Q
\end{array}
\]

from a line \( l(Q) = l_X \) in \( T \) such that \( T \rightarrow \mathbb{P}^2 \) is a conic bundle with discriminant curve \( Q \). The inverse image of 1 in \( \mathbb{P}^4 \) by this projection gives the required hyperplane section of \( T \).

This suggests that maybe one should look for the family of cubic surfaces over a level moduli space \( \tilde{\mathcal{Q}}_{5, i} \) \((i \geq 2)\). One might be able to produce it over \( \tilde{\mathcal{Q}}_{5, 2} \) or \( \tilde{\mathbb{P}}^2 \).

1.6 - The branch locus of \( P \) in \( \mathcal{Q}_5 \) and

the branch locus of \( \tilde{h} \)

We need to introduce the Abel-Jacobi mapping \( AJ \) and the Fano variety (intersection of two translates of \( \Theta \)) associated to a double solid.

The intermediate jacobian \( JZ \) can be identified with the group of algebraic one-cycles in \( \tilde{Z} \), which are algebraically equivalent to 0, modulo rational equivalence. Fixing a curve \( C_0 \) of degree \( d \) in \( \tilde{Z} \) we define (up to translation) the Abel-Jacobi mapping \( AJ \) on the Hilbert Scheme of curves of degree \( d \) in \( \tilde{Z} \) by sending a curve \( C \) to the rational equivalence class of \( C - C_0 \).

Alternatively, we can define \( AJ(C) \) to be the linear form on \( H^{2,1}(Z) \) which to a differential form \( \omega \) associates \( \int_{\Omega} \omega \) where \( \Omega \) is an \( \mathbb{R} \)-three-dimensional singular chain such that its boundary \( \partial \Omega \) is equal to \( C - C_0 \) (this is well-defined modulo \( H_3(Z, \mathbb{Z}) \)).
Let $E_Z$ be the image under $AJ$ of the Fano variety of lines of $Z$. Using the computation of the homology class of $E_Z$ in $A$ in [C1], Beauville shows that $E_Z = \Theta \cdot \Theta_x$ for an $x \in \cap \Sigma(X_i, \eta_i)$ (see the beginning of the introduction).

Recall that $\tilde{h}$ is the lift of the $\Gamma_{00}$-map to the blow up $\tilde{A}$ of $A$ at 0. Let $E$ be the exceptional divisor in $\tilde{A}$. We have the following description of the branch locus of $\tilde{h}$:

The branch locus of $\tilde{h}$ in $(\mathbb{P}^4)^*$ is the union of the dual hypersurface $T^*$ of $T$ and an irreducible hypersurface $R_0 = \tilde{h}(E)$ of $(|2\Theta|_{00})^*$ whose inverse image $R$ in $A$ is the union of the diagonals of the surfaces $\Sigma(X)$ for $X \in \mathbb{P}^1(A)$.

Moreover if $E$ is the variety of $a \in A$ such that $\Theta \cdot \Theta_a$ is the Fano variety of lines $E_Z$ of a double solid $Z$ then the inverse image of $T^*$ in $A$ is the union of $E$ and another irreducible component $R'$ whose fibers over $T^*$ have cardinality 64. The inverse image of a generic element of $T^*$ has cardinality 96. For $a \in R'$, $\Theta \cdot \Theta_a$ contains 21 curves, 6 of them counting twice.

From this we derive a second proof (there is an unpublished proof of this in [Do3]) of the fact that the branch locus of $P$ in $\mathcal{A}_5$ is the variety of intermediate jacobians of double solids; showing that extensions $A_x$ with $x \in R'$ are generalized intermediate jacobians of double solids.

1.7 - Schottky

The Schottky problem is the problem of characterizing jacobians in $\mathcal{A}_4$. In relation with that we prove:

If $A$ is not in $\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11}$ then the only base point of $|2\Theta|_{00}$ is 0 with multiplicity $2^8$.

Here $\mathcal{J}_4$ is the locus of jacobians, $\overline{\mathcal{J}}_4$ is its closure in $\mathcal{A}_4$, and $\mathcal{A}_{n11}$ is a six-dimensional subvariety of $\mathcal{A}_4$ which we will define later (see section 7).

Let $V(\Gamma_{00})$ be the base locus of $|2\Theta|_{00}$. Let $\mathcal{A}_{4\text{dec}}$ be the locus of decomposable
ppav's. Together with the result of [W3] this gives:

Let $A$ be in $\mathcal{A}_4 \setminus (\mathcal{A}_{n11} \cup \mathcal{A}_{4dec})$ then $A$ is a jacobian if and only if $V(\Gamma_{00})$ contains another point besides 0, in that case $V(\Gamma_{00}) = C - C \cup \{ \pm (K_C - 2g_3) \}$ (C-C = \{p-q \mid p,q \in C\}).

From the computation of the multiplicity (in a sense which will be made precise later) at 0 of $|2\Theta|_{00}$ we deduce:

If $A$ is not in $\mathcal{A}_4 \cup \mathcal{A}_{n11}$, then the linear system $|2\Theta|_{00}$ of quartics in $\mathbb{P}^3 = \mathbb{P}T_0A$ obtained by taking the quartic tangent cones at 0 of the elements of $|2\Theta|_{00}$ has (projective) dimension at least 3 and its base locus $V_{\inf}(\Gamma_{00})$ is empty.

Together with the result of [BD2], this gives:

Let $A$ be in $\mathcal{A}_4 \setminus (\mathcal{A}_{n11} \cup \mathcal{A}_{4dec})$ then $A$ is a jacobian if and only if $V_{\inf}(\Gamma_{00})$ is nonempty. In that case, if the unique quadric containing the canonical curve is smooth, $V_{\inf}(\Gamma_{00})$ is the canonical curve or, if the quadric containing the canonical curve is not smooth, $V_{\inf}(\Gamma_{00})$ is the union of the canonical curve and the vertex of the quadric containing it.

1.8 - Jacobians, one theta-null and singular cubic threefolds

Using the (generic) description of the cubic threefold as the hypersurface containing pencils of $\Gamma_{00}$-divisors associated to Prym curves and as the dual of a well-specified component of the branch locus of the $\Gamma_{00}$-map we can see that the cubic threefold can be defined for all elements of $\mathcal{A}_4 \setminus (\mathcal{A}_4 \cup \mathcal{A}_{n11})$ and also for jacobians of irreducible nonhyperelliptic curves.

(1.8.1) So the question arises: for which abelian varieties is the cubic threefold singular?

The answer is

The cubic threefold is singular exactly on the locus $\mathcal{A}_4$ of jacobians and on the divisor $\Theta_{null}$ of ppav's with one vanishing theta-null, i.e., those abelian varieties whose
theta divisor contains a point of order 2 which is then a singular point of $\Theta$.

When the cubic threefold $T$ has one ordinary double point and is otherwise smooth the intermediate jacobian of $T$ is the jacobian of a nonhyperelliptic curve $C$ of genus 4 [CG]. The generalized intermediate jacobian $\tilde{J}T$ of $T$ is an extension of $JT$ by $C^*$ ($\tilde{J}T$ is not the generalized jacobian of a curve by [Co]). The point $\mu$ of order 2 on $\tilde{J}T$ corresponding to $JC$ projects to 0 in $JT$. Hence $\mu$ is the only nontrivial point of order 2 in $C^* \subset \tilde{J}T$. The even points of order 2 on $\tilde{J}T$ which occur for ppav's with one vanishing theta-null do not project to 0 in $JT$.

We have a finite rational map

$$\theta_{\text{null}} \to \mathcal{S}_4$$

By the above the degree of this map is

$$2^4(2^3+1) - 1 + 2^4(2^3+1) - 1.$$  

(1.8.2) What happens to the family of double solids $Z_A$ with intermediate jacobian a fixed ppav $A$ when the cubic threefold becomes singular?

On $\theta_{\text{null}}$ the family $Z_A$ breaks up into two components: one of them, say $(Z_A)_0$, is blown down to the singular point of $T$ and the other, say $(Z_A)_1$, dominates $T$.

When $T$ has one ordinary double point and is otherwise smooth the Fano variety of lines $F$ of $T$ has a double curve which parametrizes lines through the double point of $T$, the desingularization of $F$ is the second symmetric product $C^{(2)}$ of $C$ [CG]. The one-dimensional component of the branch locus of the double cover $P^1(A) \to F$ is the double curve of $F$. The component $(Z_A)_0$ parametrizes double solids whose discriminant curves verify $(X_i,\eta_i) = \lambda(X_i,\eta_i) = (X'_i,\eta'_i)$.

It is proved in [C1] that when the branch locus of the double solid is a quadric in the quadrics through the double points $p_1,\ldots,p_6$ of $Z$ the intermediate jacobian $JZ = A$ has a vanishing theta-null. These are precisely the elements of $(Z_A)_0$. 
The component \((\mathcal{Z}_A)_1\) parametrizes double solids with determinantal branch locus.

In the jacobian case Donagi proves [Do2] that the intermediate jacobians of double solids \(Z\) with "unodal" branch locus are jacobians. A unode counts as three ordinary double points and has local equation

\[
x^2 + (y - az)(y - bz)(y - cz) + \text{arbitrary higher order} = 0
\]

The tangent cone at a unode is twice a line and the cubic term of the equation corresponds to three infinitely near double points in that line. The incidence configuration of the exceptional curves in the resolution of a unode is given by the Dynkin diagram \(D_4\):

![Dynkin diagram D_4](image)

It is easy to see that the converse is true, i.e., when \(JZ\) is a jacobian then the branch locus of \(Z\) has a unode. Double solids with two unodes correspond to elements of \(\Theta_{\text{null}} \cap D_4\).

In the jacobian case, there are no double solids above the singular point \(t\) of \(T\).

1.9 - Torelli for quartic double solids with six nodes

We are going to describe a geometric way of recovering each element of \(\mathcal{Z}_A\) (see 1.7) from the abelian variety \(A\). Suppose \(A\) generic in \(Q_4\).

(1.9.1) Recall that we have a finite surjective map \(\mathcal{Z}_A \to T\) where \((T, \mu)\) is the cubic threefold with point of order 2 associated to \(A\). The degree of this map is 32 and we describe its fibers below (Donagi first proved this on the threefold \(T\) which he associated to \(A\) in a different way: we will write down his construction later).

Let \(Z \in \mathcal{Z}_A\) and denote by \(p_1, \ldots, p_6\) its double points. Let \(\rho : \mathbb{P}^3 \to \mathbb{P}^3\) be the rational map of degree 2 defined by the linear system of quadrics in \(\mathbb{P}^3\) containing the
Denote by $\tau$ the birational involution on $\mathbb{P}^3$ commuting with $\rho$. Also for each set $\mathcal{F}$ of cardinality 4 contained in $\{p_1, \ldots, p_6\}$ we denote by $\tau_{\mathcal{F}}$ the birational involution of $\mathbb{P}^3$ defined by the linear system of cubics in $\mathbb{P}^3$ with double points at the elements of $\mathcal{F}$. These are Cremona transformations of $\mathbb{P}^3$ (see [DO]). Hence we can talk about the subgroup $\mathcal{G}$ of the Cremona group of $\mathbb{P}^3$ generated by $\{\tau_{\mathcal{F}}\}_{\mathcal{F}}$. The group $\mathcal{G}$ contains the birational involution $\tau$ and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$.

If $B$ is the branch locus of $Z : \mathbb{P}^3 \to \mathbb{P}^3$, then we denote by $\tau Z, \tau_{\mathcal{G}} Z$ the double solids with respective branch loci $\tau B, \tau_{\mathcal{G}} B$. These double solids are all birationally equivalent and have the same intermediate jacobian $A$.

The double solids $Z$ and $\tau Z$ clearly have the same double points. Also, for all $\mathcal{F}$, the double points of $\tau_{\mathcal{G}} Z$ have the same moduli in $\mathbb{P}^3$ as the $p_i$'s. Letting $\mathcal{G}$ act on $Z$, we obtain a set of 32 double solids which breaks naturally into the union of 16 subsets $\{Z_i, \tau Z_i\}$ ($1 \leq i \leq 16$) with, for instance, $Z = Z_1$ (this is not canonical and depends on the choice of $Z$).

We show that the set of double solids $Z'$ such that $D_Z = D_{Z'}$ is equal to $\{Z_i, \tau Z_i; 1 \leq i \leq 16\}$.

(1.9.2) Next, the moduli space of six points in $\mathbb{P}^3$ is birationally equivalent to the moduli space of six points in $\mathbb{P}^1$ which is birationally equivalent to the moduli space $\mathcal{M}_{g_2}$ of curves of genus 2 via the following:

Through 6 generic points in $\mathbb{P}^3$ there passes a unique twisted cubic curve so we obtain 6 points in $\mathbb{P}^1$ by identifying the cubic with $\mathbb{P}^1$. To the 6 points in $\mathbb{P}^1$ there is associated a unique (up to isomorphism) curve $M$ of genus 2 which is the double cover of the projective line ramified at the six points. The Kummer variety $K = JM/\pm id$ is the branch locus of $\rho$.

As the tangent space to $A$ at 0 can be canonically identified with

$$H^0(\mathbb{P}^3, \mathcal{O}(2) \otimes \mathcal{G})^*$$
where $\mathfrak{d}$ is the ideal sheaf of the points $p_i$ [Cl]. $K$ lives in the projectivised tangent space $\mathbb{P}T_0A$. The 16 double points of $K$ are the images by $\rho$ of

- the 15 lines through $p_i$ and $p_j$
- the twisted cubic $C_Z = C_{tZ}$ through $p_1, \ldots, p_6$

Each $Z_i$ picks a double point of $K$. If $i \neq 1$ this is the image of the line $\langle p_j, p_k \rangle$

if $Z_i = t_{\mathfrak{d}}Z$ or $t_\sigma tZ$ with

$$\mathfrak{d} = \{p_1, \ldots, p_6\} \setminus \{p_j, p_k\}$$

Under $t_{\mathfrak{d}}$, the line $\langle p_j, p_k \rangle$ goes to $C_{Z_i}$. The double point associated to $Z = Z_1$ is the image of $C_Z$.

(1.9.3) It is proven in [C2] that the projectivised tangent cone at 0 to $D_Z$ is $\rho(B)$ (see 2.1).

Let $\tilde{\mathbb{P}}^3$ be the blow up of $\mathbb{P}^3$ at the points $p_1, \ldots, p_6$. Under the lift of $\rho$ to $\tilde{\mathbb{P}}^3$, the images of the six exceptional planes are six planes $P_i$ in $\mathbb{P}T_0A$. The double point corresponding to $C_Z$ is contained in all the $P_i$'s. The intersection $P_i \cap P_j$ also contains the double point $\rho(\langle p_i, p_j \rangle)$ for $i \neq j$. Each $P_i$ contains six double points of $K$ that are on a conic. The moduli of the six points on the conic is equal to the moduli of the points $p_1, \ldots, p_6$. Hence $M$ is the double cover of the conic branched along the double points of $K$.

Each $P_i$ contains two other distinguished conics: these are the images of the projectivised tangent cones at $p_i$ to $B$ and $tB$. The intersection $\rho(B) \cap P_i$ is the union of these conics.

As $D_Z = D_{Z'}$ for all $Z' \in \{Z_i, tZ_i; 1 \leq i \leq 16\}$, we have $\rho(B) = \rho(tB) = \rho(B_i) = \rho(tB_i)$ for all $i$.

From the above discussion we deduce that there are 16 well-determined positions of $K$ with respect to $\rho(B)$ which come from the double solids. As $\rho: \mathbb{P}^3 \to \mathbb{P}^3$ is the double cover branched along $K$, once we are given $K$ and $D_Z$ (hence $\rho(B)$), we can
construct the 32 double solids $Z_i, \tau Z_i$.

Also, as the moduli of six double points of $K$ on a conic gives the isomorphism class of $M$, $K$ is determined by its double points.

All we need to do now is, given $D_Z$, find the double points of $K$. This is done below.

(1.9.4) We show that for each $i$, the images under $AJ$ of the Fano varieties of lines of $Z_i$ and $\tau Z_i$ are equal (up to translation and multiplication by $-1$), we denote them by $E_i$. For $i \neq j$, $E_i$ and $E_j$ are not images of each other by $\pm id$ or any translation.

Generically, $T$ is birationally equivalent to its dual $T^*$. For $Z \in \mathcal{Z}_A$, the $\Gamma_{00}$-divisor $D_Z \subset T$ associated to $Z$ corresponds to a hyperplane $H_Z$ in $(\mathcal{Z}_{\Theta(0)})^*$, tangent to $T^*$ at a point $t_Z$. The inverse image of $H_Z$ in $\tilde{\mathcal{A}}$ by $\tilde{h}$ is, of course (the strict transform of) $D_Z$.

Let $\pm x_i$ ($1 \leq i \leq 16$) be the elements of $A$ such that $\Theta, \Theta_{x_i} = E_i$ (see above).

As we saw in 1.2, $T^*$ is one component of the branch locus of $\tilde{h}$. By 1.6, above $t_Z, \tilde{h}$ is ramified exactly at the points $\pm x_i$ ($1 \leq i \leq 16$). We show that (after identification of $T_{\pm x_i}A$ with $T_0A$ by translation by $\pm x_i$) the kernel of the differential of $\tilde{h}$ at $\pm x_i$ is the double point of $K$ corresponding to $Z_i$.

1.10 - Theta-identities

We will also draw theta identities between an element of $\Gamma_{00}$ defining $D_Z$ and three elements of $\Gamma$ determined by the Fano variety of lines in $Z$. 
2. GENERALITIES AND BACKGROUND

Here we write down some unpublished results of Beauville, Clemens, Debarre and Donagi.

2.1 - The $\Gamma_{00}$-divisor $D_Z$ associated to a quartic double solid $Z$

We write down the construction of Clemens [C3]:

Let $\tilde{\pi}$ and $\pi$ be the projections $\tilde{Z} \rightarrow \mathbb{P}^3$ and $Z \rightarrow \mathbb{P}^3$. Consider the Hilbert scheme of irreducible conics in $\tilde{Z}$, i.e., curves $C$ in $\tilde{Z}$ such that

- $C.\tilde{\pi}^*\mathcal{O}_{\mathbb{P}^3}(1) = 2$
- $C.q_i = 0$ for all $i$, where $q_i$ is the exceptional quadric in $\tilde{Z}$ above $p_i$.

This Hilbert scheme has dimension 4 and has two components say $\mathcal{O}_Z$ and $\mathcal{O}_Z'$. The generic element of $\mathcal{O}_Z'$ is the inverse image of a line in $\mathbb{P}^3$, hence it is a smooth elliptic curve. The generic element of $\mathcal{O}_Z$ is one component of the inverse image in $\tilde{Z}$ of a conic in $\mathbb{P}^3$ which is everywhere tangent to the branch locus $B$ of $Z$. So the generic element of $\mathcal{O}_Z$ is a smooth rational curve. Clemens defines $D_Z = \text{AJ}(\mathcal{O}_Z)$, where $\text{AJ} : \mathcal{O}_Z \rightarrow JZ$ is the Abel-Jacobi map with base curve the inverse image in $\tilde{Z}$ of a line in $\mathbb{P}^3$ which is tangent to $B$ (The family of such conics is blown down to a point by $\text{AJ}$).

(2.1.1) THEOREM: The divisor $D_Z$ is an element of $|2\Theta|_{00}$. The tangent cone to $D_Z$ at 0 is $\rho(B) : the\ image\ of\ B\ by\ the\ map\ \rho\ given\ by\ the\ linear\ system\ of\ quadrics\ through\ the\ double\ points\ of\ Z$. 
Proof: Let $C$ be a conic in $\mathbb{P}^3$, everywhere tangent to $B$ and generic for this property. Let $V$ be the plane containing $C$. The data of $C$ everywhere tangent to the canonical curve $S = B \cap V$ of genus 3 is equivalent to the data of a divisor class on $S$ with square $(K_S)^2$ or equivalent to the data of a point $\gamma$ of order 2 on $JS$. Then $|K_S+\gamma|$ is a pencil. Thus there is at least a pencil of curves rationally equivalent to $C$ in $\tilde{Z}$. We want to find the reducible curves, i.e., the unions of two lines (bitangent to $S$) in this pencil. By [DO], odd theta-characteristics on $S$ are in one-to-one correspondence with bitangent lines to $S$ in $V$. A simple computation shows that there are 12 theta-characteristics $k$ on $S$ such that $k$ and $k+\gamma$ is odd. Thus there are $6 = 12/2$ pairs of bitangent lines to $S$ in $V$ whose unions cut divisors of $|K_S+\gamma|$ on $S$, equivalently, there are six pairs of incident lines in $\tilde{Z}$ whose sums are rationally equivalent to $C$.

It follows in particular that $D_Z$ is of dimension $\leq 3$. The following implies that $D_Z$ cannot have dimension less than 3.

Recall that $E_Z$ is the Fano variety of lines of $\tilde{Z}$. We just saw that if $I \subset E_Z \times E_Z$ is the divisor of pairs of incident lines, then $A(I) = D_Z$.

Suppose, momentarily, that $Z$ is smooth, then $A = IZ$ is a ppav of dimension 10.

By [W1] (p. 77) if

$$\phi : E_Z \times E_Z \to A$$

is induced by addition, then $\phi^*\Theta$ is homologous to $I$ modulo $\text{Pic}E_Z \oplus \text{Pic}E_Z$. By [W1] (page 70) the restrictions of $\phi^*\Theta$ to two fibers $l \times E_Z$ and $E_Z \times l$ have homology class $3D_1$, where $D_1 = I(l \times E_Z)$ or $(E_Z \times l)I$ respectively. The $D_1$ all have the same homology class $D$ in $E_Z$. Hence $\phi^*\Theta$ is homologous to

$$I + 2(D \times E_Z) + 2(E_Z \times D)$$

A straightforward computation, using Pontrjagin product and

$$(A) \quad \Theta^{(d-j)}/(d-j)! = \sum \gamma_{i_1} \times \delta_{i_1} \cdots \gamma_{i_j} \times \delta_{i_j}$$
(where \( j \) is any number between 1 and \( d = \dim A = 10 \) in this case, \( \gamma_i, \delta_i \) form a symplectic basis of \( H_1(A; \mathbb{Z}) \), \( \{i_1, \ldots, i_j\} \) is any set of distinct integers between 1 and \( d \) and "\( \times \)" is Pontryagin product) shows then that the homology class of \( \phi_* \mathcal{I} \) in \( A \) is \( 25.\Theta^7/\mathcal{I}! \). The map \( 1 \to AJ(I) \) is of degree at least 12. So the reduced image of \( I \) has homology class \( n.\Theta^{7}/\mathcal{I}! \) where \( n = 1 \) or 2. Now, using [BC] and degenerating one double point at a time to the case where \( Z \) has 6 double points, one sees that \( D_Z \) has homology class \( n.\Theta \) where \( n = 1 \) or 2.

At this point, Clemens observes that the tangent cone to \( D_Z \) at 0 is \( \rho(B) \) (we give a proof below). So \( D_Z \) has a singularity of order \( \geq 4 \) at the origin. As \( A \) is generic and \( \Theta \) nonsingular, \( n \) has to be 2 (since, by 3.6, \( \mid 2\Theta \mid_{00} \) has no member with a singularity of order \( > 4 \) at 0, the singularity of \( D_Z \) is of order exactly 4). That is \( D_Z \in \mid 2\Theta \mid_{00} \). (In the case where \( Z \) has less than six double points also \( n = 2 \) and the tangent cone at 0 to \( D_Z \) is \( \rho(B) \).) The proof is as follows:

Let \( C \) and \( V \) be as before. Let \( C' \) and \( C'' \) be the two components of the inverse image of \( C \) in \( \tilde{V} = \tilde{\pi}^{-1}(V) \). Let \( |C'| \) be the linear system of \( C' \) in \( \tilde{V} \). Using the identification \( \mathbb{P}T_0 A \cong \mathcal{O}(2) \otimes \mathcal{I}^* \) (see 1.9), the projectivised tangent space to \( D_Z \) at \( C' \) corresponds to the unique quadric \( Q \) through the double points of \( Z \) such that either

\[
N \subset \tilde{\pi}^*Q \quad \text{for some} \quad N \in |C'|
\]

or

\[
N.\tilde{\pi}^*Q = N.B \quad \text{for some} \quad N \in |C'|
\]

here \( \tilde{\pi} : \tilde{Z} \to \mathbb{P}^3 \) is the projection. Let \( B' \) be an element of the pencil spanned by \( B \) and \( \tilde{\pi}^*Q \) which contains some element \( N \) of \( |C'| \). Let \( l \) be a line in \( \mathbb{P}^3 \), tangent to \( B \) at a point \( t \). The curve \( \tilde{\pi}^*l \) has an embedded point at \( t \). Going to the limit, one sees that when \( N = \tilde{\pi}^*l \), \( B' \) has a singular point at \( t \). Hence, \( B \) and \( Q \) are tangent at \( t \). Hence, \( \rho(B) \) and the limits (at 0) of tangent planes to \( D_Z \) are tangent. Hence, \( \rho(B) \) is the projectivised tangent cone at 0 to \( D_Z \).
2.2 - How to recover $B$ from the plane representation of $X_i$

The curves $X_i$ (see 1.2) are also discriminant curves for the projections of $B$ from $p_i$. For all $i$, $X_i$ is a plane sextic with five double points corresponding to $p_j$ for $j \neq i$. Each plane sextic $X_i$ admits an everywhere tangent conic $C_{ii}$: this is the image of the projectivised tangent cone to $B$ at $p_i$ under $r_i$. Here $r_i$ is the extension of the projection from $p_i$ to the blow up $\mathbb{P}^3_i$ of $\mathbb{P}^3$ at $p_i$. The six points of contact of $C_{ii}$ with $X_i$ are the images by $r_i$ of the six lines through $p_i$ which have contact of order 4 with $B$ at $p_i$. A result of Donagi is [Do2]:

(2.3.1) **Theorem**: Let $B$ be a generic quartic in $\mathbb{P}^3$ with six double points in general position. Then the plane representation of $X_i$ determines the conic $C_{ii}$ which, together with the plane representation, determines $B$, hence also $Z$.

**Proof**: One first proves the uniqueness of the everywhere tangent conic.

Let $\tilde{B}$ be the blow up of $B$ at its double points. Notice that $\tilde{B}$ is the minimal desingularization of $B$ and that $\tilde{B}$ is a K3 surface. Donagi proves

(2.3.2) **Lemma**: Generically, the Picard number of $\tilde{B}$ is 6. The Picard group of $\tilde{B}$ is generated (over $\mathbb{Z}$) by the exceptional curves $E_i$ above $p_i$ and the inverse image $H$ of the hyperplane class in $\mathbb{P}^2$. These verify (for all $i$ and $j \neq i$)

$$H^2 = 4, \quad (E_i)^2 = -2, \quad H.E_i = E_i.E_j = 0$$

Assume the lemma for a moment. Suppose that there exists another everywhere tangent conic to $X_1$, say $C$. The inverse image of $C$ in $\tilde{B}$ has two components, say $C'$ and $C''$. Let $p: \tilde{B} \to \mathbb{P}^2$ be the projection. The following numerical computations yield a contradiction:

a) $C'^2 = -2$ by adjunction.

b) $C'.(H - E_1) = 2$

This is because $H - E_1$ is the line bundle associated to $p$ and $C'.(H - E_1)$ is the degree of $p_*C' = C$ in $\mathbb{P}^2$. 
c) $0 \leq E_2, C' \leq 4$

Because both curves lie over distinct conics in $\mathbb{P}^2$.

d) $C'.E_i \geq 0$ for all $i$.

By the lemma we can write

$$C' = aH + \sum b_j E_j$$

for some $a, b_j \in \mathbb{Z}$

then the above become

a) $4a^2 - 3\Sigma b_j^2 = -2$

b) $4a + 2b_1 = 2$

c) $-2 \leq b_1 \leq 0$

d) $b_i \leq 0$ for all $i$.

From b) and c) one obtains $a = 1$ and $b_1 = -1$. Then by a) exactly two of the $b_j$'s are nonzero for $j \neq 1$, say 2 and 3, and $b_2^2 = b_3^2 = 1$. Hence $C' = H - E_1 - E_2 - E_3$ and $p_\ast C'$ is twice a line.

**Proof of the lemma**: The assertion about the intersection numbers of $H$ and the $E_i$'s is clear. By [BPV] Chapter 8, for every K3 surface $B$, $H^2(B, \mathbb{Z})$ equipped with the quadratic form given by cup-product is isomorphic to the lattice

$$L = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H$$

where $E_8$ is the free $\mathbb{Z}$-module of rank 8 equipped with the quadratic form (on the canonical basis)

$$q_8(x_1, \ldots, x_8) = 3\Sigma_{1 \leq i \leq 8} x_i^2 - 2x_1x_3 - 2x_2x_4 - 3\Sigma_{3 \leq i \leq 7} x_i x_{i+1}$$

and $H$ is $\mathbb{Z}^2$ equipped with the hyperbolic quadratic form (on the canonical basis)

$$q_h = 3x_1x_2$$

Denote the quadratic form on $L$ (or cup-product on $H^2(B, \mathbb{Z})$) by $(\cdot, \cdot)$. Let $L_\mathbb{C}$

$$L \otimes \mathbb{C}$$

and for $x \in L_\mathbb{C}$ let $[x]$ be its image in $\mathbb{P}L_\mathbb{C}$. If

$$\Omega = \{ [\omega] \in \mathbb{P}L_\mathbb{C} \mid (\omega, \omega) = 0 \text{ and } (\omega, \omega) > 0 \}$$

one can define the period map which to each K3 surface $B$ with a fixed isomorphism
\( H^2(B,\mathbb{Z}) \cong L \) associates the line in \( L_\mathbb{C} \) spanned by the class of a nonzero (nowhere vanishing) differential on \( B \). We denote the period of \( B \) by \([\omega_B] \in \mathbb{P}L_\mathbb{C}\). Then by [BPV] the image of the period map is \( \Omega \).

Let \( \{e_1; 1 \leq i \leq 8\}, \{e_1; 9 \leq i \leq 16\}, \{f_1, f_2\}, \{f_3, f_4\}, \{f_5, f_6\} \) be the respective canonical basis for the first and second \(-E_8\) summands of \( L \) and the three \( H \) summands of \( L \). Consider the sublattice \( L' \) of \( L \) generated by \( e_1, e_2, e_5, e_7, e_9, e_{10}, f_1+2f_2 \). Then the symmetric bilinear form on \( L \) has the same values on our distinguished set of generators of \( L' \) as the intersection pairing on \( H, E_1 \) (with \( H \) corresponding to \( f_1+2f_2 \)). Let \( L^\perp \) be the orthogonal complement of \( L' \) with respect to the bilinear form on \( L \). Notice that as \( L' \) is a primitive sublattice of \( L \) (i.e., \( L/L' \) has no torsion) we have \( L'^\perp = L' \).

By [Sh] Chapter IX (p. 216), if \( B \) is a K3 surface corresponding to a generic element of \( \Omega \) orthogonal to \( L' \), then the algebraic part of \( H^2(B,\mathbb{Z}) \) is equal to \( L'^\perp = L' \).

Given a generic K3 surface \( \tilde{B} \) whose period is orthogonal to \( L' \), map \( \tilde{B} \) to \( \mathbb{P}^2 \) using the linear system \( |H-E_1| \): this linear system is easily seen to be of degree 2 and dimension 2. This map blows \( E_i \) down to a point for \( i \neq 1 \). So if we denote by \( B' \) the surface obtained from \( \tilde{B} \) by blowing \( E_1 \) down to points for \( i \neq 1 \), \( B' \) is a double cover of \( \mathbb{P}^2 \) branched along a plane sextic with five nodes. The image of \( E_1 \) in \( \mathbb{P}^2 \) is an everywhere tangent conic to this sextic. For dimension reasons we see that \( B' \) has exactly five ordinary double points as singularities.

Given the plane sextic \( X (= X_1) \) with five ordinary double points and everywhere tangent conic \( C_1 = C_{11} \), one recovers the quartic surface \( B \):

Let \( p' \) be the projection \( B' \to \mathbb{P}^2 \). The inverse image of \( C_1 \) in \( B' \) has two components \( C' \) and \( C'' \). By the following, \( B \) is the image of \( B' \) in \( \mathbb{P}^3 \) by the linear system \( 1 = |p'^*h+C'| \) where \( h \) is the hyperplane class in \( \mathbb{P}^2 \). (I only collapses \( C' \) to a
double point of $B$, and $B'$ has five double points hence $B$ has six double points)

a) The degree of 1 is 4. Indeed,

$$(p^*h'C')^2 = 2h^2 + C^2 + 2h.C_t = 6 + C^2$$

and

$$8 = (C'+C'')^2 = 2C'^2 + 2C'.C'' = 2C'^2 + 12.$$  

b) The dimension of 1 is 3.

The inverse image of a generic line in $\mathbb{P}^2$ is a curve $N$ of genus 2. As $p^*h.C' = 2$, $N \cup C'$ has arithmetic genus 3. The claim follows by the Riemann-Roch formula, the genus formula and the cohomology sequence associated to the exact sequence

$$0 \rightarrow \mathcal{O}_B \rightarrow \mathcal{O}_B(N) \rightarrow \mathcal{O}_{N}(N) \rightarrow 0.$$  

2.3 - The Fano surface $E_Z$ is an intersection of translates of $\Theta$

This is due to Beauville [C3].

Let $X$ be a curve in $\mathbb{P}_1(A)$. Parametrize $A$ and $\Theta$ with $X$ (see [M2] and [M3]), take

$$A = \{D : \pi_*D \equiv K_X, h^0(D) \text{ even}\} \subset \text{Pic}^8(\tilde{X})$$

$$\Theta = \{D : h^0(D) > 0\} \subset A$$

Let $Z$ be a double solid such that $X = X_1$ for $Z$ (see 1.9). Let $g_0^2 = \lfloor K_X - p - q \rfloor$ be the linear system on $X$ associated to the plane representation of $X$ given by $Z$. Consider a line 1 in $\tilde{Z}$. Then, looking at the image of 1 in $\mathbb{P}^2$, one sees that 1 picks a divisor $D_1$ in $g_0^2$ which is the sum of the lines through $p_1$ in $Z$ which are incident to 1.

Let $p', p'', q', q''$ be the liftings of $p, q$ in $\tilde{X}$. By [B3] there is a choice of these liftings, say $p', q'$, such that $h^0(X, D_1 + p' + q')$ is even as well as $h^0(X, D_1 + p'' + q'')$. So we obtain an embedding $E_Z \rightarrow \Theta.\Theta_{p''+q''-p'-q'} = \Theta.\Theta_{[p'', q'']}$

$$1 \rightarrow D_1 + p' + q'$$
By [C1] the homology class of $E_Z$ is $\Theta^2$, so we have equality.

### 2.4 - Embeddings of Prym curves in $\Theta$

We prove a fact from [C3]:

Let $(X, \eta)$ be an element of $P_5$ such that $X$ is smooth and nonhyperelliptic. Let $A = P(X, \eta)$. Let $Y$ be a curve in $P^1(A)$, tetragonally related to $X$. As in 2.3 parametrize $A$ and $\Theta$ with $Y$:

$$A = \{ D : \pi_*D \equiv K_Y, h^0(D) \text{ even} \} \subset \text{Pic}^8(\tilde{Y})$$

$$\Theta = \{ D : h^0(D) > 0 \} \subset A$$

Let $g^1_4 \in W^1_4(Y)$ relate $Y$ to $X$, then $\tilde{X}$ is one component of

$$\{ D \in \tilde{Y}(4) : \pi_*D \equiv g^1_4 \}$$

**THEOREM:** Each Prym-embedding of $\tilde{X}$ in $\Theta$ is given by

$$\tilde{X}_E = \{ D + E : D \in \tilde{X} \}$$

for $E \in \tilde{X}'$ where $\tilde{X}'$ is one of the two components of the set

$$\{ E \in \tilde{Y}(4) : \pi_*E \equiv K_Y - g^1_4 \}$$

**Proof:** Parametrize $A$ and $\Theta$ with $X$. Then a copy of $\tilde{X}$ in $A$ is of the form $\tilde{X}_E = \{ p - \sigma p + E : p \in \tilde{X} \}$ for some divisor $E$ on $\tilde{X}$ of degree 8, such that $\pi_*E \equiv K_X$ and $h^0(E)$ is odd. Therefore by Clifford's theorem $h^0(E) = 1$ or 3 because $X$ is nonhyperelliptic. As

$$\tilde{X}_E \cap \Theta = \{ p : h^0(p - \sigma p + E) > 0 \}$$

it is immediately seen that $\tilde{X}_E$ is in $\Theta$ if and only if $h^0(E) = 3$. So the set of Prym-embeddings of $\tilde{X}$ in $\Theta$ is

$$\tilde{X}' = \{ E \in \text{Pic}^8\tilde{X} : \pi_*E \equiv K_X \text{ and } h^0(E) = 3 \}.$$  

Notice that $\tilde{X}' \subset \tilde{X}''$. By [B3] the homology class of $\tilde{X}'$ is $[\Theta]^3/3$. By [W2] (see also [B]) $\tilde{X}''$ is a smooth curve for $A$ generic and its homology class is also $[\Theta]^3/3$. Hence they are equal if $A$ is generic.
In general, choose a generic point $p \in \tilde{X}$. Then (see [T]) the map $E \mapsto E+p-\sigma p$ defines an isomorphism of $\tilde{X}''$ with

$$W_p = \{ D : h^0(D - p) > 0 \} \subset \Theta.$$ 

As $p$ is generic we see that the proof of proposition 1 and the remark following it in [BD1] apply: $W_p$ is a curve with homology class $[\Theta]^3_3$.

Hence $\tilde{X}' = \tilde{X}''$.

2.5 - Embeddings of Prym curves in intersections of two translates of $\Theta$

We write down the result of [De]:

Consider a tetragonally related triple of smooth, nonhyperelliptic and nontrigonal curves $X, Y, U$ in $P^1(A)$. Let $g^1_4 \in W^1_4(Y)$ relate $Y$ to $X$ and $U$. If $Y$ is bielliptic, suppose that $g^1_4$ is not the pullback of a $g^1_2$ on an elliptic curve. Then for each $[p,q]$ and $[p,r]$ in $\Sigma(Y)$ such that $p$ and $q$ are not images of each other by a bielliptic involution of $Y$ and such that there exists $x \in \tilde{Y}$ with $\pi p + \pi q + \pi r + \pi x \equiv e^1_4$:

**PROPOSITION:** The pairs $[p,q]$ and $[p,r]$ are also elements of $\Sigma(X)$ and $\Sigma(Z)$. The intersection

$$\Theta, \Theta_{[p,q]}, \Theta_{[p,r]} = S_{pqr} \cup W_p$$

is the union of translates of $\tilde{X}', \tilde{Y}', \tilde{U}'$ with the notations of 2.4.

**Proof:** By [BD1] we have:

$$\Theta, \Theta_{[p,q]}, \Theta_{[p,r]} = S_{pqr} \cup W_p$$

Where $S_{pqr}$ is defined to be

$$S_{pqr} = \{ D : h^0(D - p - q - r) > 0 \} \subset \Theta.$$

The homology classes of $W_p$ and $S_{pqr}$ are respectively $[\Theta]^3_3$ and $2[\Theta]^3_3$ by [BD1] and [B3].

We have
\(|K_Y - \pi p - \pi q - \pi r | = \pi x + |K_Y - g^4_d|
\)
as linear systems and by the definition of the tetragonal construction in 1.3, \(S_{pqr}\) splits as the union of
\[p+q+r+x+\tilde{X}' \quad \text{and} \quad p+q+r+\sigma x+\tilde{U}''\]
where \(X', U'\) are the two curves tetragonally related to \(Y\) via \(K_Y - g^4_d\). Now, using 3.2 (we do not use 2.5 to prove 3.2), it is easy to see that \([p,q]\) and \([p,r]\) are also elements of \(\Sigma(X)\) and \(\Sigma(Z)\).

2.6 - Prym-embeddings of \(\tilde{X}_{ij}\) in \(E_Z\)

This is due to Clemens.

Proposition: For each \(i, j\) there are (two) Prym-embeddings of \(\tilde{X}_{ij}\) in \(E_Z\).

Proof: As we noted in the introduction, for all \(i \neq j\), \(X_i, X_j\) and \(X_{ij}\) are tetragonally related. The \(g^4_d\) on \(X_1\) is the one given by lines through the image of \(p_j\) in the plane representation of \(X_1\). Take \(i = 1, j = 2\). Let \(l_i\) be a line through \(p_2\), corresponding to a point \(t \in X_2\). Then \(l_i\) projects to a line through the image of \(p_2\) in \(\mathbb{P}^2\) and picks (via incidence) a lifting, say \(s_1+s_2+s_3+s\) of a divisor in \(l_{g_6^2} - p_2^\prime - p_2^\prime\prime\), where \(p_2^\prime\) and \(p_2^\prime\prime\) are the two points of \(X_1\) above \(p_2\). The pair of incident lines \((l_s, l_t)\) through \(p_1\) and \(p_2\) corresponding to \((s,t) \in X_{12} \subset X_1 \times X_2\) picks, via incidence, the divisor \(s_1+s_2+s_3+\sigma s\) in \(l_{g_6^2} - p_2^\prime - p_2^\prime\prime\).

Recall that an "actual" line in \(\tilde{Z}\) is, for instance, the union of \(l_s\) and a ruling of \(Q_1\) (this is analogous to the case of conics, see 2.1). However, we will not need to take this into account in our computations.

Let \(V\) be the plane spanned by \(\tilde{\pi}(l_s)\) and \(\tilde{\pi}(l_t)\). Then \(\tilde{V} = \tilde{\pi}^{-1}(V)\) is a Del Pezzo surface isomorphic to \(\mathbb{P}^2\) blown up at 7 points \(q_i\) \((0 \leq i \leq 6)\) such that \(q_0, q_1, q_2\) and \(q_3, q_4, q_5, q_6\) are colinear. The plane \(V\) is the image of \(\tilde{V}\) by the map defined by the linear system of strict transforms of cubics in \(\mathbb{P}^2\) passing through the \(q_i\)'s. The lines \(\langle q_0,q_1 \rangle\)
and \((q_0, q_3)\) are blown down to \(p_1\) and \(p_2\). By [Dm] a basis of the Picard group of \(\tilde{V}\) is given by

\[H, -E_0, \ldots, -E_6\]

where \(H\) is the pullback of the hyperplane class in \(\mathbb{P}^2\) and \(E_i\) is the exceptional divisor above \(q_i\).

When the \(q_i\)'s are in general position, the lines in \(\tilde{V}\) are given by:

- exceptional curves \(E_i\) (7 curves)

and the strict transforms of the following curves in \(\mathbb{P}^2\):

- lines through 2 of the \(q_i\)'s (21 curves)
- conics through 5 of the \(q_i\)'s (21 curves)
- cubics through all the \(q_i\)'s and with a double point at one of them (7 curves).

In our degenerate case, some of the above curves coincide.

If, for instance, \(l_s = L_{35}\) (\(L_{iq}\) is the strict transform of the line through \(q_i\) and \(q_j\)), and \(l_t = E_3\), we have

\[l_s + l_t = H + (-E_5) = L_{56} + E_6 = E_0 + L_{05}\]

Notice that \(E_0\) and \(L_{56}\) are the two lines in \(\tilde{V}\) which project to \((p_1, p_2)\) in \(\mathbb{P}^3\).

So

\[l_{st} = l_s + l_t - L_{56} = E_6\]

and \((l_{st})' = l_s + l_t - E_0 = L_{05}\)

are lines in \(\tilde{Z}\). Similarly, the same holds for the other pairs of incident lines. Thus we obtain two Prym-embeddings of \(\tilde{X}_{12}\) in \(E_Z\) \((l_{s'}, l_{t'}) \mapsto l_{st}\) and \((l_{s'}, l_{t'}) \mapsto (l_{st})'\).
3. PRELIMINARIES

We prove some preliminary results that we will need later on.

Throughout this section, unless otherwise stated, $A$ is a generic element of $A_4^0$.

(3.1) Let $(\tilde{X},X)$ and $(\tilde{Y},Y)$ be two smooth elements of $P^{-1}(A)$. Define $\tilde{X}_\lambda$ to be the curve parametrizing Prym-embeddings of $\tilde{X}$ in $\Theta$. By 2.4, $\tilde{X}_\lambda$ has a fixed point free involution $\sigma_\lambda$. Let $X_\lambda$ be the quotient of $\tilde{X}_\lambda$ by $\sigma_\lambda$. By 2.4, if $Y$ is tetragonally related to $X$ through $g_4^1$ on $Y$, then $Y$ is tetragonally related to $X_\lambda$ through $K_X - g_4^1$. Hence $X_\lambda$ is of genus $5$ and $P(\tilde{X}_\lambda,X_\lambda) = A$. Hence if we choose a double solid (as in the introduction) such that, with the notations of the introduction, $X = X_1$, then $X_\lambda = X'$ (take $Y = X_2$). From this it becomes clear that $\lambda : (\tilde{X},X) \mapsto (\tilde{X}_\lambda,X_\lambda)$ is an involution. Also $(\tilde{X}_\lambda,X_\lambda) \neq (\tilde{X},X)$ in general. This can be seen for instance by doing the same construction for $A = JC$ the Jacobian of a smooth, irreducible curve (see 4.2, notice that we do not use this in 4.2).

(3.2) PROPOSITION: The intersection $\Theta . \Theta_\lambda$ contains a Prym-translate of $\tilde{X}_\lambda$ if and only if $x \in \Sigma(X)$. In that case, $\Theta . \Theta_\lambda$ contains exactly two copies of $\tilde{X}_\lambda$ which correspond to $s, t$ where $x = [s,t]$.

Proof: Looking back at 2.4 we see that we have a surface in $\Theta$ traced by the Prym-embeddings of $\tilde{X}$ (or $\tilde{X}_\lambda$) in $\Theta$. Considering $A$ naturally embedded in $J\tilde{X}$ (resp. $J\tilde{X}_\lambda$) these embeddings are translates of

$$\{ p - \sigma p ; p \in \tilde{X} \}$$

(resp. $$\{ p - \sigma p ; p \in \tilde{X}_\lambda \}$$)

in $J\tilde{X}$ (resp. $J\tilde{X}_\lambda$). Abusing notations, we denote by $p - \sigma p$ an element of $A$ which is
the translate of \( p \cdot \sigma \in \hat{X} \) (resp. \( J\hat{X}_\lambda \)) by a fixed element of \( J\hat{X} \) (resp. \( J\hat{X}_\lambda \)).

So the above surface is the set of elements \( a \cdot p \cdot \sigma + q \cdot \sigma q \) for \( p \in \hat{X}, q \in \hat{X}_\lambda \) and some fixed \( a \in A \).

Suppose that we have an embedding of \( \hat{X}_\lambda \) in \( \Theta \cdot \Theta_x \) for \( x \in A \).

Then there exists \( p \in \hat{X} \) such that

\[
a \cdot p \cdot \sigma p + q \cdot \sigma q \in \Theta \cdot \Theta_x \text{ for all } q \in \hat{X}_\lambda
\]

that is

\[
a \cdot p \cdot \sigma p + q \cdot \sigma q \cdot x \in \Theta \text{ for all } q \in \hat{X}_\lambda
\]

This implies that there exists \( p' \in \hat{X} \) such that

\[
a \cdot p \cdot \sigma p + q \cdot \sigma q \cdot x = a \cdot p' \cdot \sigma p + q \cdot \sigma q
\]

so we obtain \( x = p \cdot \sigma p \cdot p' + \sigma p' \in \Sigma(X) \) and in fact we have exactly 2 embeddings of \( \hat{X}_\lambda \) in \( \Theta \cdot \Theta_x \).

Conversely, for all \( x \in \Sigma(X) \) and \( q \in \hat{X}_\lambda \) the Prym-translate of \( \hat{X} \) in \( \Theta \) corresponding to \( q \) and its translate by \( x \) intersect in two points which trace two copies of \( \hat{X}_\lambda \) in \( \Theta \cdot \Theta_x \) as \( q \) varies.

Notice that, with the notations of 2.4 and 2.5, the Prym-translates of \( \hat{X}_\lambda \) in \( \Theta \cdot \Theta_{[s,t]} \) for \( [s,t] \in \Sigma(X) \) are just \( W_s \) and \( W_t \).

Q.E.D.

(3.3) **Lemma**: Suppose that \( Z \) is a double solid such that \( X = X_1 \) is obtained by projection from one of its double points \( p_1 \) (cf 2.2), then \( \Sigma(X) \subset D_Z \).

**Proof**: We first notice that \( \Sigma(X) \) is equal to the image by \( AJ \) (with base point the inverse image in \( \tilde{Z} \) of a line in \( \mathbb{P}^3 \) tangent to \( B \)) of

\[
\{ l_p \cup L^1_p \cup L^2_q \cup l_q \ ; p, q \in \hat{X} \}
\]

where \( l_p, l_q \) are the strict transforms in \( \tilde{Z} \) of the lines in \( Z \) corresponding to \( p, q \); \( L^1 \), \( L^2 \) are the two rulings of the exceptional quadric \( Q_1 \) in \( \tilde{Z} \) above \( p_1 \) and \( L^1_p \) is the line of the ruling \( L^1 \) which passes through \( Q_1 \cap l_p \). By definition, a conic in \( \tilde{Z} \) is a curve \( C \) verifying

\[- C.\pi^* \Theta(1) = 2 \]
- $\mathcal{Q}_i = 0$ for all $i$.

The restriction of the ideal sheaf of $\mathcal{Q}_1$ to $\mathcal{Q}_1$ is $\mathcal{O}_{\mathcal{Q}_1}(-1,-1)$. So to obtain an actual conic in $\tilde{Z}$ with respect to $\tilde{\pi}^*\mathcal{O}(1)$ we have to add the two rulings of $\mathcal{Q}_1$ to $l_p \cup l_q$. Also

$$\text{AJ}(l_p \cup L_p^1 \cup L_q^2 \cup l_q) = \text{AJ}(l_p \cup L_p^2 \cup L_q^1 \cup l_q).$$

This is because both surfaces are equal to the union of the Prym-embeddings of $\tilde{X}$ which pass through the origin.

Let $V$ be the plane spanned by $\tilde{\pi}(l_p)$, $\tilde{\pi}(l_q)$ in $\mathbb{P}^3$ and define $L_{pq} = \mathcal{Q}_1 \cap \tilde{\pi}^{-1}(V)$. Then $L_{pq}$ is a $(1,1)$-curve in $\mathcal{Q}_1$. Hence

$$\text{AJ}(l_p \cup L_p^1 \cup L_q^2 \cup l_q) = \text{AJ}(l_p \cup L_{pq} \cup l_q).$$

We claim that $l_p \cup L_{pq} \cup l_q$ is in the linear equivalence class of a smooth rational conic in $\tilde{\pi}^{-1}(V)$ except when $p = \sigma q$. In the latter case, the image of $l_p \cup L_{pq} \cup l_{\sigma p}$ is twice a line and

$$\text{AJ}(l_p \cup L_{pq} \cup l_q) = 0$$

with the convention of 2.1.

Indeed, as in 2.6, $\tilde{V} = \tilde{\pi}^{-1}(V)$ is a Del Pezzo surface isomorphic to the blow up of $\mathbb{P}^2$ at 7 points $q_i$. The 7 points are not in general position but, as $\tilde{\pi}^{-1}(V)$ has a double point, 3 of them, say $q_1, q_2, q_3$, lie on a line which is precisely $L_{pq}$. The arithmetic genus of $C = l_p \cup L_{pq} \cup l_{\sigma p}$ is 0. The map $\tilde{V} \to V \cong \mathbb{P}^2$ is given by the anticanonical bundle $-K_{\tilde{V}}$ of $\tilde{V}$ [Dm]. Hence, as the image of $C$ is a conic, we have

$$-K_{\tilde{V}} \cdot C = 2.$$

The genus formula then gives $C.C = 0$. Using Riemann-Roch and the cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_{\tilde{V}} \to \mathcal{O}_{\tilde{V}}(C) \to \mathcal{O}_C(C) \to 0$$

one sees that $C$ moves in a linear system of (projective) dimension 1. The arithmetic genus of $C$ being 0, the generic member of this pencil is a smooth rational conic. Q.E.D.
(3.4) **LEMMA**: With the notations of 1.9, for a generic double solid $Z$ and any discriminant curve $X_{ij}$ for $Z$:

$$\Sigma(X_{ij}) \subseteq D_Z$$

**Proof**: As in 2.6, let $(l_u,l_v)$ represent an element of $\tilde{X}_{12}$. Then, with the notations of the proof of 3.3,

$$\{ \text{AJ}(L_u^i \cup l_u \cup l_v \cup L_v^j) ; (l_u,l_v) \in \tilde{X}_{12} \}$$

is a Prym-embedding of $\tilde{X}_{12}$ in $D_Z$ and hence in $A$, for all $i,j \in \{1,2\}$. As before the base point for the Abel-Jacobi mapping is the inverse image in $\tilde{Z}$ of a line in $\mathbb{P}^3$ tangent to $B$. Thus, still with the notations of the proof of 3.3, a translate of

$$\{ \text{AJ}(l_s \cup l_t \cup l_u \cup l_v \cup L_{su} \cup L_{tv}) ; (l_s,l_t), (l_u,l_v) \in \tilde{X}_{12} \}$$

is equal to $\Sigma(X_{12})$. Actually, with our choice of base point, this is exactly $\Sigma(X_{12})$ because it is the union of the Prym-embeddings of $\tilde{X}_{12}$ that pass through the origin.

The image of $l_s \cup l_t \cup l_u \cup l_v \cup L_{su} \cup L_{tv}$ by the Abel-Jacobi mapping is equal to the rational equivalence class of

$$l_s + l_t + l_u + l_v + L_{su} + L_{tv} - l_s - l_{is} - L_s^1 - L_s^2 - l_t - l_{it} - L_t^1 - L_t^2$$

or, as $L_{su} - L_s^1 - L_s^2$ and $L_{tv} - L_t^1 - L_t^2$ are rationally equivalent to 0, to

$$l_u + l_v - l_{is} - l_{it} = l_u + l_v - l_{is} - l_{it}$$

or, by 2.6, to

$$l_{uv} - l_{st} = l_{uv} + l_{st} - l_{st} - l_{st}$$

For $(u,v)$ and $(s,t)$ generic the lines $l_{uv}$ and $l_{st}$ do not intersect each other so $\text{AJ}(l_{uv} + l_{st})$ is not an element of $D_Z$.

Q.E.D.

Recall that in 1.9 we introduced the (Cremona) group $\mathcal{G}$ of birational transformations of $\mathbb{P}^3$ and described its orbits in $\mathcal{Z}_A$ (see notations). We have the following :

(3.5) **PROPOSITION**: Let $\{Z_1, vZ_1, \ldots, Z_{16}, tZ_{16}\}$ be the orbit under $\mathcal{G}$ of a generic element $Z_1$ of $\mathcal{Z}_A$. Then

i) $E_{Z_j} = E_{Z_j}$ for all $j$ and
ii) \( E_j \neq E_k \) for all \( j \neq k \).

(By equality (resp. inequality) we mean that they are (resp. are not) images of each other by an automorphism of \( A \)).

\textbf{Proof:} The map \( \rho : \mathbb{P}^3 \to \mathbb{P}^3 \) given by the linear system of quadrics through the nodes \( p_i \) of \( Z \) is the composition:

\[
\mathcal{O}(2) \\
\mathbb{P}^3 \dashrightarrow \mathbb{P}^9 \dashrightarrow \mathbb{P}^3
\]

where the arrow on the right is the projection from the nodes of \( Z \). \( \rho \) is a rational map of degree 2 and blows up the double points of \( Z \). So we can define the rational involution \( \iota \) which interchanges the sheets of \( \rho \).

The rational involutions \( \iota_{\mathcal{F}} \) are compositions of

\[
\mathcal{O}(3) \\
\mathbb{P}^3 \dashrightarrow \mathbb{P}^{19} \dashrightarrow \mathbb{P}^3
\]

where the arrow on the right is projection from the projective tangent spaces to the image of \( \mathbb{P}^3 \) in \( \mathbb{P}^{19} \) at the points of \( \mathcal{F} \) where \( \mathcal{F} \) is a subset of cardinality 4 of \( \{ p_1, \ldots, p_6 \} \).

i) For a line \( l \) in \( \mathbb{P}^3 \) (bitangent to \( B \) but otherwise generic) \( \iota(l) \) is a curve of degree 7 with double points at the \( p_i \)'s and elsewhere tangent to \( B \).

So we need to show that for each line \( l' \) in \( \tilde{Z} \) there is a curve \( C' \) in \( \tilde{Z} \) (isomorphic to its projection \( C \) in \( \mathbb{P}^3 \), such that \( C \) is a septic everywhere tangent to \( B \) and passes doubly through the \( p_i \)'s) such that \( l' \cup C' \) is a complete intersection in \( \tilde{Z} \).

(Then the image of \( l' \cup C' \) under \( AJ \) is constant when \( l' \) varies.)

Let \( l \) be \( \pi(l') \), the unique quadric in \( \mathbb{P}^3 \) containing \( l \) and the \( p_i \)'s. Let \( q' \neq q \) be a quadric from the pencil of those passing through the \( p_i \)'s and the two points of contact of \( l \) with \( B \).

In the pencil spanned by \( B \) and \( (q')^2 \) there exists a quartic \( B' \) containing \( l \). Let \( B, B', q' \) denote also the equations of \( B, B' \) and \( q' \). Write \( B' = B - r(q')^2 \) for a
complex number r. Then, as $B = N^2$ in $\tilde{Z}$, the equation of $B'$ in $\tilde{Z}$ is $(N - \sqrt{r} \cdot q')(N + \sqrt{r} \cdot q')$. Hence the inverse image of $B'$ in $\tilde{Z}$ is the union of two surfaces one of which, say $Q$, contains $l$. Then $Q \cap \tilde{r}^{-1}(q)$ is a complete intersection of the required type. (Similar proofs are in [C3] and [W].)

ii) Consider a subset $\mathcal{F} \subset \{p_1, \ldots, p_6\}$ of cardinality 4, say $\mathcal{F} = \{p_1, \ldots, p_4\}$ and let $Z' = t_{\mathcal{F}}Z$. The plane spanned by $p_2, p_3, p_4$ is blown down by $t_{\mathcal{F}}$ to a double point $p_1'$ of $Z'$ and lines through $p_1$ in $Z$ go to lines through $p_1'$ in $Z'$. So if $X$ is the discriminant curve for the projection of $B$ from $p_1$, then $X$ is also the discriminant curve for the projection of $B' = t_{\mathcal{F}}B$ from $p_1'$.

Projecting from $p_1$ and $p_1'$ we have two plane representations of $X$ as a sextic with five double points that are not projectively equivalent because they determine the nonprojectively equivalent quartics $B$ and $B'$ (2.2). Suppose that these plane representations are given by the linear systems $|K - p - q|, \, |K - p' - q'|$. By 2.3, for some choice of liftings of $p, q, p', q'$ say $p_1, q_1, p_1', q_1'$ in $\tilde{X}$, we have:

$$F_Z = \Theta_{[p, q]} \circ \Theta_{[p_1, q_1]}, \quad F_{Z'} = \Theta_{[p_1', q_1']}$$

as $(p, q) \neq (p', q')$ these two varieties cannot be images of each other by an automorphism of $A$.

Q.E.D.

Let $\omega_X$ be the invertible sheaf of regular differentials on $X$ and $\Omega_A$ be the rank 4 free sheaf of regular differentials on $A$. Let $S^4 = S^4H^0(A, \Omega_A) = S^4H^0(X, \omega_X \otimes \eta)$ be the space of symmetric quadrilinear forms on $T_0A$. We have a linear map

$$\tau : \Gamma_{00} \rightarrow S^4$$

which to each element of $\Gamma_{00}$ associates the quartic term of its Taylor expansion at 0.

(3.6) **PROPOSITION**: For $A = JC$ the jacobian of a nonhyperelliptic curve $C$ of genus 4 with a unique $g^1_3$ (denoted by $\zeta$), the mapping $\tau$ is injective. (A is an element of $\Theta_{\text{null}} \cap \mathcal{J}_4$)

**Proof**: For $\zeta \in \text{Pic}^3C$, let $\Theta_{\zeta} = W_3 - \zeta$. Let $\theta_{\zeta}$ be a nonzero section of $\mathcal{O}_A(\Theta_{\zeta})$. 

For \( D \in T_0A \), denote by \( D\theta_{\xi} \) be the derivative of \( \theta_{\xi} \) with respect to \( \xi \), in the direction \( D \). Let

\[
\theta_{\xi} = q + g + \ldots
\]

be the Taylor expansion of \( \theta_{\xi} \) at 0, with \( q \) and \( g \) irreducible polynomials of respective degrees 2 and 4 ([ACGH]). The quadric

\[
q \in S^2H^0(A,\Omega_A) = S^2H^0(C,\omega_C) = \text{Hom}(H^0(A,\Omega_A)^*,H^0(A,\Omega_A)) = \text{Hom}(T_0A,(T_0A)^*)
\]

has rank 3. Let \( D_0 \in T_0A \) be a generator of its kernel. Then ([BD2]) the canonical model of \( C \) is the complete intersection of \( \bar{q} = \{q = 0\} \) and \( \bar{f} = \{f = D_0g = 0\} \), the space \( T_{00} \) is generated by \( (\theta_{\xi})^2 \) and the \( DD_0\theta_{\xi}\theta_{\xi} - D_0\theta_{\xi}D\theta_{\xi} \) for \( D \in T_0A \). The quartic terms of the Taylor expansions at 0 of \( (\theta_{\xi})^2 \) and the \( DD_0\theta_{\xi}\theta_{\xi} - D_0\theta_{\xi}D\theta_{\xi} \) are respectively

\[
q^2 \quad \text{and} \quad 2(q.Df - f.Dq)
\]

If \( \tau \) is not injective, then there exist \( D \in T_0A \) and \( r \in C \) with \( (r,D) \neq (0,0) \) such that

\[
\begin{align*}
q.Df & - f.Dq = 0 \\
r.q^2 + q.Df & - f.Dq = 0
\end{align*}
\]
i.e.,

\[
q.(r.q + Df) = f.Dq
\]

As \( q \) and \( f \) are irreducible this implies

\[
r.q + Df = Dq = 0
\]

Hence \( D \) is a multiple of \( D_0 \). We can suppose \( D = D_0 \). We get

\[
r.q + D_0f = 0
\]

and applying \( D_0 \) to this equation we obtain \( (D_0)^2f = 0 \). This would imply that the image of \( D_0 \) in \( PT_0A \) is in \( \bar{q} \cap \bar{f} = C \). The ruling of \( \bar{q} \) cuts on \( C \) the divisors of \( \xi \). If the singular point of \( \bar{q} \) is on \( C \), then the ruling of \( \bar{q} \) will cut a \( g_2 \) on \( C \) and \( \kappa C \) will have a double point at the vertex of \( \bar{q} \). Both of these are impossible. Q.E.D.

(3.7) **Remark**: This result can be proved in a similar way for a curve \( C \) with two distinct
$g_3^1$s. It can also be proved for a hyperelliptic curve in a different way. However, we do not need these here.
4. INTERSECTIONS OF TWO TRANSLATES OF $\Theta$

In this section our aim is to compute the number of Prym-embedded curves in an intersection of two translates of theta.

(4.1) PROPOSITION: For $A = JC$ the jacobian of a generic curve $C$ of genus 4, a generic intersection of two translates of $\Theta$ contains translates of 27 distinct Prym-embedded curves.

Proof:

(4.2) The fiber of the Prym map at $A$ splits into two components each isomorphic to $C^{(2)}$ (the second symmetric product of $C$) (see [B1]) and exchanged by the involution $\lambda$.

One component is the space whose generic elements are trigonal curves related to $C$ via Recillas' construction ([R]). More precisely, given a $g_4^1 = K_C - p - q$ on $C$, one defines $\tilde{X}$ as the set of elements $\{s,t\}$ of $C^{(2)}$ such that $h^0(g_4^1 - s - t) > 0$. The involution $\sigma$ on $\tilde{X}$ is defined by sending $\{s,t\}$ to $\{s',t'\}$ whenever $s+t+s'+t' \in g_4^1$. The points of a divisor of the $g_3^1$ on $X = \tilde{X}/\sigma$ are the three partitions of a divisor of $g_4^1$ into two sets of two points.

The other component is the space of singular curves with one double point whose normalization is $C$. If $C_{pq}$ is the curve obtained from $C$ by identifying the two points $p$ and $q$, the double cover $\tilde{C}_{pq}$ of $C_{pq}$ with $A$ as Prym variety is obtained by taking two copies of $C$ and identifying $p$ on one copy with $q$ on the other.

(4.3) View $A$ as $\text{Pic}^3 C$ and take $\Theta = W_3$ (the effective divisor classes). Then for each trigonal $X$ we see immediately that we have the following embeddings of $\tilde{X}$ in $\Theta$ (and
only these):

\[ \{s,t\} \mapsto s+t+p_0 \]

or by \[ \{s,t\} \mapsto K_C - (s+t+p_0) \quad \text{for all } p_0 \in C. \]

As the family of the above embeddings is \( \tilde{C}_{pq} \), we deduce that \( X_\lambda = C_{pq} \).

Analogously, the embeddings of \( \tilde{C}_{pq} = C_1 \cup_{p=q} C_2 \) (\( C_1 \cong C \)) are given by:

\[ C_1 \ni s \mapsto s_1 + s_2 + s \]

\[ C_2 \ni t \mapsto K_C - t - t_1 - t_2 \quad \text{for all } s_1, s_2, t_1, t_2 \text{ such that } s_1 + s_2 + t_1 + t_2 \equiv g_4 \equiv K_C - p - q. \]

(4.4) We have a surjective morphism of \textit{generic} degree 6:

\[ C^{(2)} \times C^{(2)} \rightarrow A \]

\[ (a+b, a'+b') \mapsto a+b - a'-b' \]

If (generically) \( a+b - a'-b' = c+d - c'-d' \) for distinct elements of \( C^{(2)} \times C^{(2)} \), then

\[ a+b+c'+d' \equiv c+d+a'+b' \equiv K_C - p - q \quad \text{for some } p, q \in C. \]

So the number of pairs \( (c+d, c'+d') \) with this property (for fixed \( a, b, a', b' \)) is equal to the number of bisectant lines \( \langle p, q \rangle \) to the canonical model of \( C \) which intersect simultaneously the lines \( \langle a, b \rangle \) and \( \langle a', b' \rangle \). Equivalently, we need to have \( h^0(g - p - q) > 0 \) and \( h^0(g' - p - q) > 0 \) where \( g = K_C - a - b \) and \( g' = K_C - a' - b' \).

The preceding inequalities define two correspondences on \( C \times C \) and the number we are looking for is half the intersection number of these correspondences. By [GH] (p. 285) these correspondences are homologous to \( 4E + 4F - \Delta \) where \( E \) and \( F \) are fibers of the two projections \( C \times C \rightarrow C \) and \( \Delta \) is the diagonal. The intersection number is \( (4E + 4F - \Delta)^2 = 33.6.8E.\Delta - 8.8E.\Delta + \Delta^2 = 10 \) (as \( \Delta^2 = -(2g-2) = -6 \)), half of it is 5 and the degree is 5+1 = 6.

(4.5) We will now compute the number of Prym-embedded curves in \( \Theta.\Theta_\lambda \) for \( x \) generic in \( A \). We will first consider trigonal curves. Our notations will be as in 4.2 and 4.3.
The set of effective divisors in $\text{Pic}^1C$ is one-dimensional ($= C$) and, for $x$ generic, $C+x$ will not intersect it. Hence we can assume $h^0(p_0+x) = 0$ for all $p_0 \in C$. Equivalently $h^0(K_C - p_0-x) = 2$. Let $g_5^1 = K_C - p_0-x$. By 4.3, we need to know when for all $(s,t) \in \tilde{\mathcal{X}}$ we will have

1) $h^0(s+t+p_0+x) > 0$ or

2) $h^0(K_C - s-t - p_0+x) > 0$.

By Riemann-Roch, case 2) is equivalent to case 1) with $x$ replaced by $-x$. By Riemann-Roch again, case 1) is equivalent to:

For all $s,t$, if $h^0(g_4^1 - s-t) > 0$, then $h^0(g_5^1 - s-t) > 0$.

This is possible if and only if $g_5^1 = g_4^1 + t_0$ for some $t_0$ in $C$, or if and only if $t_0+p_0+x = K_C - g_4^1 = q_0+s_0$ is an effective divisor.

Taking into consideration the cases 1 and 2, we see that for each of the 6 representatives $a_i+b_i-a_i^\prime-b_i^\prime$ ($1 \leq i \leq 6$) of $x$ in $C^{(2)} \times C^{(2)}$ we have (two) embeddings of the trigonal curves obtained from $K_C - a_i-b_i$ and $K_C - a_i^\prime-b_i^\prime$ in $\Theta.\Theta_x$.

Looking at (4.4) we see that for all $i \neq j$ there is a line $l_{ij}$ bisecant to the canonical model of $C$ which encounters simultaneously $\langle a_i,b_i \rangle, \langle a_i^\prime,b_i^\prime \rangle, \langle a_j,b_j \rangle$ and $\langle a_j^\prime,b_j^\prime \rangle$. Hence the number of lines incident to $\langle a_i,b_i \rangle, \langle a_i^\prime,b_i^\prime \rangle$ for some $i$ (or incident to $\langle a_j,b_j \rangle, \langle a_j^\prime,b_j^\prime \rangle$ for some $i, j$) is 15.

By 4.3, in the singular case, $\tilde{C}_{pq}$ embeds in $\Theta.\Theta_x$ if and only if for some $s_1,s_2,t_1,t_2$ such that $s_1+s_2+t_1+t_2 \equiv g_4^1 = K_C - p - q$:

$h^0(t+t_1+t_2+x) > 0$ and $h^0(K_C - t-s_1-s_2+x) > 0$ for all $t \in C$. Equivalently:

$h^0(t_1+t_2+x) > 0$ and $h^0(s_1+s_2-x) > 0$.

So we must have $t_1+t_2 = a_i^\prime+b_i^\prime, s_1+s_2 = a_j+b_j$ and $(p,q) = l_{ij}$ for some $i \neq j$.

Counting everything we obtain 27 curves in $\Theta.\Theta_x$. Q.E.D.

(4.6) PROPOSITION: A generic element of $A$ lies on exactly 27 surfaces $\Sigma(X)$.

Proof: Let $\mathcal{H}$ be the Siegel half space of symmetric $4 \times 4$ complex matrices whose
imaginary part is positive definite. On $\mathcal{X}$ we have a universal family of abelian varieties $\mathcal{A}$ and a universal theta divisor defined by Riemann's theta function. Let $\mathcal{C}_A \subset \mathcal{A} \times \mathbb{P}^1(A)$ be the reduced variety of all pairs $(a,X)$ such that $a \in \Sigma(X)$. Pulling back we have a diagram:

$$\cup_A \mathcal{C}_A = \mathcal{C} \subset \mathcal{E} \rightarrow \mathcal{R} \rightarrow \mathbb{P}_5$$

$$\begin{array}{c}
\mathcal{R} \downarrow \quad \mathcal{P} \downarrow \quad \mathcal{P} \downarrow \quad \mathcal{P} \downarrow \\
\mathcal{A} \rightarrow \mathcal{X} \rightarrow \mathcal{A}_4
\end{array}$$

The number we are interested in is the generic degree of $p$. Using lemma 4.1 we need to show that $p$ is unramified at a generic jacobian.

The cotangent space to $\mathbb{P}_5$ at $(X,\eta)$ is canonically isomorphic to $H^0(X,\omega_X^2)$ [B1]. We have the following exact sequences of cotangent spaces at $(a,X) \in \mathcal{C}$:

$$0 \rightarrow H^0(X,\omega_X^2) \rightarrow T_{(a,X)}^{*} \mathcal{C} \rightarrow T_a^{*} \Sigma(X) \rightarrow 0$$

$$0 \rightarrow S^2H^0(X,\omega_X \otimes \eta) \rightarrow T_{(a,X)}^{*} \mathcal{A} \rightarrow H^0(X,\omega_X \otimes \eta) \rightarrow 0$$

We have a surjection: $H^0(X,\omega_X \otimes \eta) \twoheadrightarrow T_a^{*} \Sigma(X)$ and multiplication: $S^2H^0(X,\omega_X \otimes \eta) \hookrightarrow H^0(X,\omega_X^2)$ (i.e., the Prym map is unramified [B1]).

Hence at a generic abelian variety we have an isomorphism between $H^0(X,\omega_X^2)$ and the kernel of the composition $T_{(a,X)}^{*} \mathcal{A} \rightarrow H^0(X,\omega_X \otimes \mu_X) \rightarrow T_a^{*} \Sigma(X)$. Or an isomorphism between the kernel $N$ of $H^0(X,\omega_X \otimes \mu_X) \rightarrow T_a^{*} \Sigma(X)$ and $H^0(X,\omega_X^2) / S^2H^0(X,\omega_X \otimes \mu_X)$. This corresponds to the deformations of $X$ in $\mathbb{P}T_0A$ and means that $X$ moves in two directions transversal to the line $\langle p,q \rangle$ where $a = [p,q] \in \Sigma(X)$. Equivalently the deformations of $X$ (i.e., the curves in $\mathbb{P}^1(A)$) fill $\mathbb{P}T_0A$ (because $a$ can take any generic value in $\Sigma(X)$).

These remain valid when $A = JC$ is a generic jacobian. For $C_{st}$ a singular curve, the Prym-canonical embedding of $C_{st}$ is the union of the canonical model of $C$ and the line $\langle s,t \rangle$ ([B2]). A generic quadric in $\mathbb{P}T_0A$ does not contain $\langle s,t \rangle$, hence the Prym
mapping is still unramified. The lines \( (s,t) \) fill \( \mathbb{P}^3 \), hence the deformations of \( C_{st} \) fill \( \mathbb{P}^3 \). Letting \( \lambda \) act, we obtain the same result for \( X \) trigonal in \( P^1(JC) \). \( \Box \).

(4.7) COROLLARY: A generic intersection of two translates of \( \Theta \) in a generic abelian variety \( A \) of dimension 4 contains translates of 27 different Prym-embedded curves.

Proof: By (3.2), for \( a \in A \) and \( X \in P^{-1}(A) \), \( \tilde{X} \) admits an embedding as a Prym-curve in \( \Theta.\Theta_a \) if and only if \( a \in \Sigma(X_\lambda) \). Now use 4.6.

(4.8) PROPOSITION: For a double solid \( Z \) in \( \mathcal{Z} \), the Prym-embedded curves in \( E_Z \) are the

\[ \tilde{X}_i, (\tilde{X}_i)_\lambda, \tilde{X}_{ij}. \]

Proof: Clearly, we have an embedding of \( \tilde{X}_i \) in \( E_Z \) for each choice of ruling of the exceptional quadric above \( p_i \) in \( \tilde{Z} \). As \( E_Z = E_{TZ} \), we have two embeddings of \( (\tilde{X}_i)_\lambda \) in \( E_Z \). By 2.6, we also have two embeddings of \( \tilde{X}_{ij} \). So we know all the 27 curves.
5. THE CUBIC THREEFOLD AND THE $|2\Theta|_{00}$ DIVISORS

We are going to construct the cubic threefold. Throughout this section, $A$ is a generic ppav.

(5.1) To each curve $X \in P^{1}(A)$ we associate a pencil $l_{X}$ of $\Gamma_{00}$-divisors in the following way:

There is a one-dimensional family $\mathcal{Z}_{X}$ of double solids $Z$ such that $X = X_{1}$ is a discriminant curve for the projection from one of the double points $p_{1}$ of $Z$. Recall (3.1) that $X'_{1} = (X_{1})_{\lambda} = X_{\lambda}$. Looking at the action of the birational involution $\iota$, one sees that $X_{\lambda}$ is the discriminant curve for the projection of $\iota Z$ from $p_{1}$ [C3].

Consider the $\Gamma_{00}$-divisor $D_{Z}$ associated to $Z$ (2.1), then by 3.6, since $A$ is generic, the map $\tau$ is injective. As the tangent cone at $0$ to $D_{Z}$ is $\rho(B) = \rho(\iota B)$ (see 2.1) we have $D_{Z} = D_{\iota Z}$.

Hence, using 3.3, one sees that for all $Z$ as above, $D_{Z}$ contains $\Sigma(X) \cup \Sigma(X_{\lambda})$.

Using Pontrjagin product and (A) (in 2.1) we see that the homology class of $\Sigma(X)$ is $2\Theta^{2}$. Hence the set of $\Gamma_{00}$-divisors $D_{Z}$ containing $\Sigma(X) \cup \Sigma(X_{\lambda})$ is a pencil that we denote by $l_{X} = l_{X_{\lambda}}$ (no other curve $Y$ verifies $l_{Y} = l_{X}$).

Notice that for all $i \in \{1, \ldots, 16\}$, $D_{Z_{i}}$ (see 3.3) contains $\Sigma(X_{j}) \cup \Sigma(X_{j} \lambda)$ (for $j \in \{1, \ldots, 6\}$), hence $D_{Z_{i}} = D_{Z}$.

Let $T = T_{A} \subset |2\Theta|_{00}$ be the variety swept by the $l_{X}$ for $X \in P^{1}(A)$. If $D : Z \rightarrow |2\Theta|_{00}$ is the map which to a double solid $Z$ associates $D_{Z}$, $T$ is the image of $D$.

By the above, $D$ is constant on the orbits of the Cremona group (see 1.9). The following
implies (among other things) that $T$ is a threefold.

(5.2) LEMMA: For $x$ generic in $A$ the hyperplane $\tilde{h}(x) \in ((2\Theta_0)^*)$ (image by the $\Gamma_{00}$-map) contains exactly 27 lines $l_X$.

Proof: The hyperplane $\tilde{h}(x)$ is the set of $\Gamma_{00}$-divisors that contain $x$. A line $l_X$ is in $\tilde{h}(x)$ if and only if the $\Gamma_{00}$-divisors in $l_X$ contain $x$, or if and only if

$$x \in [\Sigma(X) \cup \Sigma(X_\lambda)]$$

So by 4.6, for $x$ generic, we have 27 such lines. Q.E.D.

(5.3) Remarks: 1) It can be seen directly, using the complete intersection technique of the proof of 3.5 i), that the double solids in the orbits of the Cremona group all have the same $\Gamma_{00}$-divisor of conics.

2) One can derive from 5.1 a proof of the fact that the tangent cone at $0$ to $\Sigma(X)$ is (scheme theoretically) the Prym-canonical image of $X$.

As in 1.6, let $\mathcal{E} \subset A$ be the subvariety of points $x$ such that $\Theta_\times = E_Z$ is the Fano of an element $Z$ of $Z_A$. Let $x \in \mathcal{E}$ be generic. As $D_Z$ contains $\Sigma(X_\times) \cup \Sigma((X_\times)_\lambda)$, $D_Z$ is the intersection point of the six lines $l_i$ corresponding to $X_i$. The hyperplane $\tilde{h}(x)$ contains the lines $l_i$. Generically, the $l_i$'s are not in a plane. Otherwise, the plane would contain also the $l_{ij}$'s. A hyperplane containing two of these planes contains too many lines. Hence $\tilde{h}(x)$ contains $PT_\times T$ and we have proved

(5.4) PROPOSITION: At a point $x$ such that $\Theta_\times = E_Z$ for a double solid $Z$, the hyperplane $\tilde{h}(x)$ is tangent to $T$.

In the same way as 5.2, using 4.8, we obtain:

(5.5) LEMMA: For $x \in \mathcal{E}$ generic, the hyperplane $\tilde{h}(x)$ contains exactly 21 lines $l_X$, 6 of which "count twice".

We give a list of important corollaries of these below.

(5.6) COROLLARY: Through a smooth point of $T$ only 6 lines $l_X$ pass.

Proof: First notice that, generically, by 3.4, none of the $l_{ij}$'s passes through the
intersection point of the \( l_i \)'s. If there are more than six lines \( l_i \) passing through \( D \in T \), then all these lines (and also the lines incident to two of them) have to be in the tangent space at \( D \) to \( T \). As before the tangent hyperplane to \( T \) would contain too many lines.

(5.7) **COROLLARY**: At a smooth point \( D \) of \( T \), the curves \( X_i \) and \( (X_i)_\lambda \) are the only Prym-curves whose \( \Sigma \)'s are in \( D \).

(5.8) **COROLLARY**: Two generic curves \( X \) and \( Y \) in \( \mathbb{P}^1(A) \) are tetragonally related if and only if \( l_X \) and \( l_Y \) intersect.

**Proof**: If \( X \) and \( Y \) are tetragonally related, we can find a double solid \( Z \) such that \( X = X_1 \) and \( Y = X_2 \) then \( D_Z \) contains \( \Sigma(X) \) and \( \Sigma(Y) \) and is the intersection point of \( l_X \) and \( l_Y \).

If \( l_X \) and \( l_Y \) intersect in \( D_Z \), then by 5.7, after renaming, \( X = X_1 \) and \( Y = X_2 \) are discriminant curves for \( Z \). By the proof of 2.6 and 1.9 they are tetragonally related.

(5.9) **COROLLARY**: Two generic curves \( X \) and \( Y \) are tetragonally related if and only if \( \Sigma(X) \cap \Sigma(Y) \) is of dimension 1. For any fixed \( X \) the set of curves \( Y \) such that \( X \) and \( Y \) are tetragonally related or equivalently such that \( \Sigma(X) \cap \Sigma(Y) \) is of dimension 1 is isomorphic to the inverse image \( \tilde{Q} \) of \( \tilde{Q} \) in \( \mathbb{P}^1(A) \), \( \tilde{Q} \) being the family of lines in \( T \) incident to \( l_X \). (\( \tilde{Q} \) maps 2-1 to \( W^4(X) = \text{set of } g_4 \)'s on \( X \))

**Proof**: If \( X \) and \( Y \) are tetragonally related then there exists a double solid \( Z \) such that \( X \) and \( Y \) are the discriminant curves for the covers \( B \to \mathbb{P}^2 \) when we project from two double points of \( Z \). By 3.3, \( \Sigma(X) \) and \( \Sigma(Y) \) are contained in \( D_Z \) and their intersection is of dimension one.

Conversely suppose that \( \Sigma(X) \cap \Sigma(Y) \) is of dimension 2. By 3.2 for any \( a \in \Sigma(X) \cap \Sigma(Y) \) there are Prym-embeddings of \( \tilde{X} \) and \( \tilde{Y} \) in \( \Theta, \Theta_a \). Hence, by 5.2, \( h(a) \) contains \( l_X \) and \( l_Y \) and we have a one-dimensional family of hyperplanes in \( \mathbb{P}^4 \) (the projective space containing \( T \)) which contain \( l_X \) and \( l_Y \); these two lines are in a plane so they intersect and by 5.8 \( X \) and \( Y \) are tetragonally related. \( \text{Q.E.D.} \)
(5.10) **COROLLARY**: Suppose that the \( Z_i \) and \( \tau Z_i \) (as in 3.5) are generic, then these are the only double solids with associated \( \Gamma_{00} \)-divisor \( D_Z \).

**Proof**: If \( Z \) is a double solid corresponding to \( D \), then by 5.7, the discriminant curves for \( Z \) are \( X_i \) or \( (X_i)_\lambda \) (these being the curves corresponding to the lines \( l_j \) through \( D \)).

Letting the Cremona group (see 2.1) act we can suppose that the curves \( X_i \) are the discriminant curves for the projections from the double points \( p_i \) of \( Z \). It has been proved by Donagi and Clemens that the curves \( X_2, \ldots, X_6 \) determine the plane representation of \( X_1 \). By 2.2 this implies that they determine the double solid \( Z \). Our proof is close to Donagi's.

By 2.6, the curves \( X_2, \ldots, X_6 \) are tetragonally related to \( X_1 \). Thus they determine 5 \( g_4^1 \) 's, say \( g_2, \ldots, g_6 \), on \( X_1 \). These have the property
\[
g_i + p_i + p_i'' = g_6^2
\]
where \( p_i', p_i'' \) are the points of \( X_1 \) above \( p_i \) and \( g_6^2 \) gives the plane representation of \( X_1 \) as discriminant curve for \( Z \). Hence if \( g_6^2 = K_X - p - q \), then
\[
h^0(K_X - g_i - p - q) > 0
\]

We see (cf. 1.3) that the singular quadrics \( q_i \) corresponding to the \( g_i \) 's are on a line \( 1 \) in the net of quadrics containing the canonical model of \( X_1 \). We need to show that the \( g_i \) 's determine \( \{p, q\} \) uniquely. This is a consequence of the following.

Let \( \tilde{Q}' = W_4^1 \) be the variety parametrizing the \( g_4^1 \) 's on \( X = X_1 \). Then (see for instance [ACGH]) \( J_X \) is the Prym variety of the cover \( \tilde{Q}' \to Q \). The \( g_i \) 's give a lift of the divisor \( D_1 \in g_6^2 \) cut on \( Q \) by \( 1 \).

Recall the elementary fact about quadrics (of rank 4) that a line is contained in \( q_i \) if and only if it is contained in one of the rulings of the quadric. For a quadric \( q_i \) of rank 3, a line is contained in \( q_i \) if and only if it is contained in its (unique) ruling.

Let \( S \) be the special subvariety of \( P(\tilde{Q}', Q) = J_X \) corresponding to \( g_6^2 \), i.e., a
translate of one component of the set
\[ \{ D \in \text{Pic}^5 X : h^0(D) > 0, \pi_* D \in g^2_5 \} \]
(the two components of this are isomorphic, see [B3]). For each bisecant \((s, t)\) to \(\kappa X\) we can consider the five (counted with multiplicities) quadrics of rank \((\leq)\) 4 containing \(\kappa X\) and \((s, t)\). Then \((s, t)\) will be contained in one ruling of each quadric and hence will pick a \(g^1_4\) above it. This defines a map \(X^{(2)} \to S\) which is an isomorphism because \(S\) has the minimal homology class ([B3]) and the image of \(X^{(2)}\) by this map is exactly its image by the Abel-Jacobi map.

Q.E.D.

We know the Prym-embedded curves in a Fano variety \(E_Z\) associated to a double solid \(Z\) (4.8). For a generic \(x \in A\), we want to describe the Prym-embedded curves in \(\Theta_\Theta_x\).

(5.11) **Lemma**: Suppose that \(X\) and \(Y\) (smooth) are tetragonally related by \(g^1_4\) on \(X\). Then
\[ \Sigma(X) \cap \Sigma(Y) = \left\{ [p, q] : h^0(g^1_4 - \pi p - \pi q) > 0 \right\} \]

**Proof**: Let \(S = \left\{ [p, q] : h^0(g^1_4 - \pi p - \pi q) > 0 \right\} \subset \Sigma(X)\) and denote by the same symbol the corresponding subvarieties of \(XXX, X^{(2)}\) and \(X^{(2)}\).

As in 4.4, the homology class of \(S\) in \(XXX\) is \(4E + 4F - \Delta\). So the homology class of \(S\) in \(X^{(2)}\) is \(8F + 8E - \Delta - \tilde{\Delta}'\) (\(\tilde{\Delta}\) is the diagonal, \(\tilde{\Delta}'\) is the set of \((p, \sigma p)\) and \(E, F\) are the two fibers in \(X^{(2)}\)). Hence in \(A\), as the map from \(S \subset X^{(2)}\) to \(A\) is of degree 2 onto its image (still denoted by \(S\)), the (reduced) homology class of \(S\) is
\[ \frac{1}{2} (16\Theta^3 / 3 - 4\Theta^3 / 3) = 2\Theta^3 \]
(the embedding of a fiber in \(A\) gives us a Prym-embedding of \(X\) with homology class \(\Theta^3 / 3\), \(\tilde{\Delta}\) has \(2^2 = 4\) times the homology class of a Prym-embedded copy of \(X\) because multiplication by 2 has degree 4 and \(\tilde{\Delta}\) has dimension 1, finally \(\tilde{\Delta}'\) is blown down to 0).

From the fact that \(\Sigma(X) \cup \Sigma(X_{\lambda})\) is the base locus of the pencil of \(\Gamma_{00}\)-divisors
we deduce that the homology class of \( \Sigma(X) \cap \Sigma(Y) \) is also \( 2\Theta^3 \). By 2.5, \( S \subset \Sigma(X) \cap \Sigma(Y) \), hence we have equality.

(5.12) **Lemma**: If three generic lines \( l_X, l_Y, l_U \) are in a plane section of \( T \), then either \( \{X,Y,U\} \) or \( \{X_\lambda, Y_\lambda, U_\lambda\} \) is a tetragonal triple.

**Proof**: By 5.8, \( X \) is tetragonally related to \( Y \) and \( U \). Let \( g_1 \) and \( g_2 \) be the \( g_4 \)'s relating \( X \) to \( Y \) and \( U \) respectively. The one-dimensional family of hyperplanes in \( 2\Theta l_{00} \) is the image by \( \tilde{h} \) of

\[
(\Sigma(X) \cup \Sigma(X_\lambda)) \cap (\Sigma(Y) \cup \Sigma(Y_\lambda)) \cap (\Sigma(U) \cup \Sigma(U_\lambda))
\]

The intersection of this with \( \Sigma(X) \) is

\[
\Sigma(X) \cap (\Sigma(Y) \cup \Sigma(Y_\lambda)) \cap (\Sigma(U) \cup \Sigma(U_\lambda))
\]

Let

\[
S_i = \{ [p,q] ; h^0(g_i - \pi p - \pi q) > 0 \} \subset \Sigma(X)
\]

and

\[
T_i = \{ [p,q] ; h^0(K_X - g_i - \pi p - \pi q) > 0 \} \subset \Sigma(X)
\]

By 5.11 and 3.1, the above is equal to \( (S_1 \cup T_1) \cap (S_2 \cup T_2) \). This being of pure dimension 1, we must have

either \( S_1 = S_2, T_1 = T_2 \)

or \( S_1 = T_2, T_1 = S_2 \).

In the first case, \( g_1 = g_2 \) and \( \{X,Y,U\} \) is a tetragonal related triple. In the second case, \( g_1 = K_X - g_2 \) and \( \{X,Y,U_\lambda\} \) is a tetragonal related triple; changing the \( g_4 \) on \( U_\lambda \) to its opposite, one sees that this is equivalent to \( \{X_\lambda, Y_\lambda, U_\lambda\} \) is a tetragonal related triple.

Q.E.D.

(5.13) Let \( a \) be a generic element of \( A \). Let \( \tilde{X}_\lambda \) be a Prym-embedded curve in \( \Theta \cdot \Theta_a \), then by 3.2 \( a = [p,q] \) for some \( p, q \in \tilde{X} \). If \( g_i \) (\( i = 1, \ldots, 5 \)) are the 5 \( g_4 \)'s such that

\[
h^0(g_4 - \pi p - \pi q) > 0
\]

(as in the proof of 5.10) and \( (Y_i, U_i) \) is the pair of curves tetragonally related to \( X \) via \( g_i \), then by 2.5 and 3.2 we have embeddings of \( (\tilde{Y}_i)_\lambda \), \( (\tilde{U}_i)_\lambda \) in \( \Theta \cdot \Theta_a \); this gives us 11 curves in \( \Theta \cdot \Theta_a \) and 11 lines in \( H = \tilde{h}(a) \).

We claim that for a generic, no three lines of the form \( l_X \) in \( H \) pass through the
same point. Otherwise, as $H$ is not tangent to $T$, the three lines will have to be in a plane. By 5.12, the three lines correspond to three tetragonally related curves $Y, Y', Y''$. Considering the hyperplane tangent to $T$ at the intersection point of $l_Y, l_{Y'}$, and $l_{Y''}$, we see that generically this is not possible (use 3.4).

Now considering $Y_1$ one can repeat the same procedure and obtain eight new (by 5.12 and the fact that no three lines in $H$ have a common intersection point) lines and Prym-embedded curves for $\Theta.\Theta_a$. Then we obtain again eight new lines and Prym-embedded curves for $\Theta.\Theta_a$ with $U_1$ instead of $Y_2$.

So we have all the Prym-embedded curves in $\Theta.\Theta_a$ together with their corresponding lines (11+8+8). Notice that, if we fix $l_X, l_Y$ and $l_U$, all the other lines corresponding to Prym-curves in any hyperplane containing them encounter one of them. Also $l_X, l_Y, l_U$ are the only lines (corresponding to Prym-curves) in their plane.

Consider $Y_2$: as before we have eight lines in four planes containing $l_{Y_2}$ and different from $l_X, l_Y, l_U$. However we have only 27 curves in $\Theta.\Theta_a$: these lines must occur already for $Y_1$ or $U_1$. By symmetry 4 of them must be in common with $Y_1$, the 4 others in common with $U_1$. So we know 5 lines in $T$ intersecting simultaneously $Y_1$ and $Y_2$. The same is true for the other pairs of lines in $H$.

We are now ready to prove:

(5.14) **Theorem**: $T$ is a cubic threefold.

**Proof**: Consider a generic tetragonal triple $\{X,Y,U\}$. Then (5.12) the lines $l_X, l_Y, l_U$ are in a plane $V$. We are going to show that $V \cap T$ is the union of these three lines, each occurring with multiplicity 1.

Suppose that there is a point $t \in V \cap T$ which is not on $l_X \cup l_Y \cup l_U$. Let $l$ be a line through $t$, corresponding to some Prym-curve. Let $H$ be the hyperplane spanned by $1$ and $V$. Then by 5.13 all lines (corresponding to Prym-curves) in $H$ encounter one of the lines $l_X, l_Y, l_U$. Hence $l$ is in $V$. However $V$ contains only $l_X, l_Y, l_U$
(5.13).

Take a point \( t \in l_X \setminus (l_Y \cup l_U) \). The tangent space to \( T \) at \( t \) contains \( l_X \), five other lines through \( t \) and the lines incident to two of these (5.5). None of these 21 lines is in \( V \) except \( l_X \) (5.13). If \( T_t(T) \) contains \( V \), it contains too many lines. Q.E.D.

(5.15) **PROPOSITION**: The only base point of \( |2\Theta_{l00}| \) is 0 if \( A \) is sufficiently generic.

**Proof**: Let \( S = \{ [p,q] ; h^0(g_4^1 - \pi p - \pi q) > 0 \} \) and \( T = \{ [p,q] ; h^0(K_X - g_4^1 - \pi p - \pi q) > 0 \} \), then by 5.11 \( S = \Sigma(X) \cap \Sigma(Y) \).

Suppose that \( x \in A \) is a base point of \( |2\Theta_{l00}| \) distinct from the origin. Then, for all \( X \in P^1(A) \), we must have \( x \in \Sigma(X) \) or \( x \in \Sigma(X_A) \). By the irreducibility of \( P^1(A) \) for \( A \) generic this implies

for all \( X \in P^1(A) \), \( x \in \Sigma(X) \)

and, in particular, choosing a generic curve \( X \) in \( P^1(A) \), \( x \in \Sigma(X) \cap \Sigma(Y) \) for all curves \( Y \) tetragonally related to \( X \). So for all \( X \in P^1(A) \):

for all \( g_4^1 \)'s on \( X \): \( h^0(g_4^1 - \pi p - \pi q) > 0 \).

Thus the line \( \langle \pi p, \pi q \rangle \) in the canonical space of \( X \) is contained in all singular quadrics containing the canonical model of \( X \).

However ([ACGJ]) the intersection of these quadrics is \( X \) unless \( X \) is trigonal in which case \( A \) is the jacobian of a curve. Q.E.D.

(5.16) **PROPOSITION**: \( T \) is a cubic threefold with at most isolated double points as singularities.

**Proof**: Suppose that the cubic threefold \( T \) has a triple point \( t \). Then \( T \) is the cone of vertex \( t \) over an irreducible cubic surface \( S \). The family of lines in \( T \) has the following components: lines through \( t \) and lines in the planes projecting to lines in \( S \). The lines \( l_X \) corresponding to Prym-curves cannot be all in a plane because otherwise the pencil of hyperplanes containing the plane will be a line in \( (2\Theta_{l00})^* \) whose inverse image in \( A \) will be contained in \( \Sigma(X) \cup \Sigma(X_A) \) for every Prym-curve \( X \). This is impossible because
then $|2\Theta|_{00}$ would have a base point besides $0$ (the lines $l_X$ generate $|2\Theta|_{00}$ by 5.14). Hence the lines $l_X$ all pass through $t$. Then the divisor corresponding to $t$ contains $\Sigma(X) \cup \Sigma(X_\lambda)$ for all $X$. However the map $\Sigma_A = (\cup \Sigma(X)) \to A$ is surjective by 4.6.

Suppose now that $T$ has a double curve $C$. Then every hyperplane section of $T$ will be singular. Also, as $T$ is irreducible, a general hyperplane section of $T$ will contain a finite number of lines and as it is singular it will contain less than 27 lines. This is ruled out by 5.2.

Q.E.D.

Consider a generic line $l_X \subset T$. The projection from $l_X$ exhibits $T$ as a conic bundle over $\mathbb{P}^2$. The discriminant curve in $\mathbb{P}^2$ for this conic bundle is the image of the set of plane sections of $T$ that are unions of three lines. By 5.12, 3.1 and the uniqueness of the plane representation of a plane quintic, this is projectively isomorphic to the plane quintic $Q$ parametrizing singular quadrics through the canonical model $\kappa X$ of $X$ (similarly for $X_\lambda$). Generically on $P^1(A)$ the discriminant curve $Q$ is smooth, so:

(5.17) **Proposition:** $T$ is smooth for a generic abelian variety.

We will determine the degree, the ramification locus and the branch locus of $\tilde{h}$ and $h$. For this, we will give a precise description of the fibers of $\tilde{h}$. This description will help us to relate directly $|2\Theta|_{00}$ to the spaces of quadrics through the canonical model of $X$. The relation between a $\Gamma_{00}$-divisor $D$ and the quadric $q$ associated to it is geometrically expressed in terms of biseccants of $\kappa X$ lying in $q$.

(5.18) **Proposition:** The degree of $\tilde{h}$ is $2^7$.

**Proof:**

(5.19) Choose a smooth curve $X \in P^1(A)$. Consider the net $N$ of quadrics containing the canonical embedding of $X$ with the plane quintic $Q$ corresponding to singular quadrics. Let $\psi : X^{(2)} \to N^* = (\mathbb{P}^2)^*$ be the morphism associating to $(p,q)$ the line $l_{pq}$ in $N$ consisting of all quadrics containing the line $\langle p, q \rangle$ in $|K_X|^{*}$. Take a line $L$ in
N which is not in the image of the diagonal by \( \psi \), is not tangent to \( Q \) and does not pass through any double point of \( Q \). Let \( Y \) and \( U \) be tetragonally related to \( X \) through two \( g_i \)'s \( g_1 \) and \( g_2 \) which project to two distinct points on \( Q \cap L \). Then, by 5.17, \( l_Y, l_U \) span a hyperplane \( H \) in \( |2\Theta|_{00} \) with \( l_X \subset H \).

(5.20) Notice that

\[
\tilde{h}^{-1}(H) = [\Sigma(Y) \cup \Sigma(Y_\lambda)] \cap [\Sigma(U) \cup \Sigma(U_\lambda)] \subset [\Sigma(X) \cup \Sigma(X_\lambda)]
\]

(see the proof of 5.2). Suppose that \( h([p,q]) = H \) with \( [p,q] \in \Sigma(X) \), then (see 5.13) the five pairs of lines incident to \( l_X \) in \( H \) project in \( N \) to the intersection points of \( l_{pq} (= l_{\pi p, \pi q}) \) with \( Q \). As the lines \( l_Y \) and \( l_U \) are among these lines we must have for \( i = 1 \) and \( 2 \):

either \( h^0(g_i, -\pi p - \pi q) > 0 \)

or \( h^0(K_X - g_i, -\pi p - \pi q) > 0 \).

That is, in the canonical space of \( X \), the line spanned by \( \pi p, \pi q \) is in the intersection \( S \) of the singular quadrics \( q_i \) corresponding to \( g_i \) for \( i = 1, 2 \).

There are 16 lines in \( S \) that are all bisecant to \( XX \). Hence we have \( 17.4 = 64 = 2^6 \) distinct points in \( \tilde{X}^{(2)} \) that project to these. By assumption 5.19, these project to \( 2^6 \) distinct points in \( \Sigma(X) \) that are distinct from \( 0 \). Counting the points in \( \Sigma(X) \cup \Sigma(X_\lambda) \) we obtain \( 3.2^6 = 2^7 \) distinct preimages for \( H \) apart from the origin.

Now the proposition follows from proposition 5.15. Q.E.D.

(5.21) DEFINITION: We define the multiplicity of \( |2\Theta|_{00} \) at \( 0 \) in the following way:

For a generic subsystem \( H \) of codimension 1 of \( |2\Theta|_{00} \) the base locus of \( H \) is a finite set. This is the union of \( 0 \) and a set of distinct points, distinct from \( 0 \). The point \( 0 \) occurs with a certain multiplicity in the intersection of the elements of \( H \): more precisely this is the length of the maximal Artinian subscheme of \( A \) with underlying set \( 0 \) and contained in every element of \( H \). This length is an upper-semicontinuous function on \( (|2\Theta|_{00})^* \) and is constant on a nonempty open subset of \( (|2\Theta|_{00})^* \). The multiplicity at \( 0 \)
for \( H \) generic is the multiplicity of \( \mathcal{I}_0 \) at 0.

(5.22) \textbf{Remark}: It is clear from the above definition that the sum of the multiplicity at 0 of \( \mathcal{I}_0 \) and the number of points distinct from 0 in the base locus of a generic \( H \) is equal to the number of points in the base locus of a generic subsystem of (projective) dimension 3 of \( \mathcal{I}_0 \). Hence the previous results imply in particular that the multiplicity at 0 of \( \mathcal{I}_0 \) is \( 4^4 \) because (see 5.2) the inverse image of a generic hyperplane \( H \) is scheme-theoretically:

\[
[\Sigma(Y) \cup \Sigma(Y')] \cap [\Sigma(Z) \cup \Sigma(Z')]
\]

the above intersection multiplicity is \( (2\theta)^4 = 2^8 \cdot 3 = 2^7 + 4^4 \).

For each \( \Gamma_0 \)-divisor \( D \), let \( B(D) \) be the projectivised tangent cone to \( D \) at 0. Recall that \( \mathcal{I}_0 \) is the linear system generated by the quartic tangent cones \( B(D) \).

(5.23) \textbf{Corollary}: If \( A \) is sufficiently generic, the base locus of \( \mathcal{I}_0 \) is empty. In particular, \( \tilde{h} \) is a morphism.

\textbf{Proof}: Consider four generic \( \Gamma_0 \)-divisors \( D_1, \ldots, D_4 \). Let \( b: \tilde{A} \to A \) be the blow up map. Let \( \tilde{D}_1 \) be the strict transform of \( D_1 \) in \( \tilde{A} \).

We have to show that \( \tilde{D}_1 \cap \ldots \cap \tilde{D}_4 \cap E \) is empty and that the \( D_i \)'s all have multiplicity 4 at 0. For this we will show that if this is not the case, then the multiplicity of \( \tilde{D}_1 \cap \ldots \cap \tilde{D}_4 \) at 0 is greater than \( 4^4 \); this contradicts 5.22.

The multiplicity at 0 of \( D_1 \cap \ldots \cap D_4 \) is given by the multiplicity of 0 in the cycle

\[
b_* [(b^*D_1 \cdot b^*D_2 \cdot b^*D_3 \cdot b^*D_4)].
\]

The \( D_i \)'s being generic, they all have the same multiplicity \( 2n \) at 0 with \( n \geq 2 \).

Writing \( b^*D_i = 2nE + \tilde{D}_i \) and developing we obtain

\[
(b^*D_1 \cdot b^*D_2 \cdot b^*D_3 \cdot b^*D_4) = \tilde{D}_2 \cdot \tilde{D}_3 \cdot \tilde{D}_4 \cdot \tilde{D}_1 \cdot \tilde{D}_i \cdot \tilde{D}_j \cdot \tilde{D}_k \\
+ \sum_{1 \leq i < j < k \leq 4} (2nE)^3 \tilde{D}_i \cdot \tilde{D}_j \cdot \tilde{D}_k \\
+ \sum_{1 \leq i \leq 4} (2nE)^4 \tilde{D}_i \\
+ (2nE)^4
\]

If \( \tilde{D}_1 \cap \ldots \cap \tilde{D}_4 \cap E \) is nonempty, then \( \tilde{D}_2, \tilde{D}_3, \tilde{D}_4, \tilde{D}_4 \) will have a positive
contribution to the multiplicity at 0, say a (because $\tilde{D}_1E = \mathcal{O}_{\mathbb{P}^3}(4)$ is positive, see [F] p. 218). Summing up the multiplicities at 0 we get
\[ a + 5(2n)^4 - 7(2n)^4 + 5(2n)^4 - (2n)^4 = a + (2n)^4 \]

By 5.22, this has to be equal to $4^4$. Q.E.D.

(5.24) As the discriminant curve for the projection of $T$ from $l_X$ is $Q$ (see 5.16), the space of hyperplanes in $l_2\Theta_{l00}$ that contain $l_X$ can be identified with the dual space of the net $N \supset Q$ parametrizing quadrics through $kX$. By the description of the fibers of $\tilde{h}$ in 5.20, the restriction of $\tilde{h}$ to $\tilde{X}^{(2)} \subset \tilde{A}$ factors through the map $\psi : X^{(2)} \to N^*$. For each pencil of quadrics $l \in N^*$, $\tilde{h}^{-1}(l)$ is the inverse image in $\tilde{X}^{(2)}$ of the bisecants to $kX$ that are in the intersection of the quadrics of $l$. Hence for each line in $N^*$, corresponding to a quadric $q$ in $N$, $\tilde{h}^{-1}(q)$ is the inverse image in $\tilde{X}^{(2)}$ of the set of bisecants to $kX$ that are in $q$. A line in $N^*$ is a pencil of lines in $N$. As we have the projection from $l_X$:
\[ l_2\Theta_{l00} \to N \]
by inverse image, a pencil of lines in $N$ will give a pencil of hyperplanes in $l_2\Theta_{l00}$ containing a fixed plane $V$ in $l_2\Theta_{l00}$ such that $V \supset l_X$. The intersection of a $\Gamma_{l00}$-divisor which is a generic element of $V$ with $\Sigma(X)$ is exactly $\tilde{h}^{-1}(q)$. This gives a geometric way of associating a quadric $q_D \in N$ to a $\Gamma_{l00}$-divisor $D$. Hence a geometric interpretation of the projection from $l_X$ onto $N$.

We are now ready to determine the branch locus of $\tilde{h}$, we will show:

(5.25) PROPOSITION: The branch locus of $\tilde{h}$ (resp. $h$) has two components: $R_0 = \tilde{h}(E)$ and $T^*$, $h^{-1}(R_0)$ is the union of the diagonals of the $\Sigma(X)$ ($X \in \mathbb{P}^{-1}(A)$).

Proof: By 5.23, $\tilde{h}$ is a morphism. Thus the branch locus of $\tilde{h}$ is a divisor. From the description of the inverse image of a hyperplane $H$ under $h$ given in 5.20 we deduce that $\tilde{h}$ can branch in three ways:

i) one of the inverse images $x$ of $H$ lies in $\Sigma(X) \cap \Sigma(X_h)$,
ii) two of the lines (bisecant to the canonical model of $X$) in the intersection of the quadrics in $l_{pq}$ ($= S = q_1 \cap q_2$ in 5.20) coincide,

iii) one of the inverse images of $H$ is in the diagonal $\Delta(X)$ of $\Sigma(X)$.

In case i) $X$ and $X_\lambda$ are Prym-embedded in $\Theta \cdot \Theta_x$. As curves correspond to lines and $X$ and $X_\lambda$ correspond to the same line, the number of lines in $T_H = T \cap H$ is less than 27 and $H$ is tangent to $T$.

In case ii) also $H$ is tangent to $T$:

The quartic Del Pezzo surface $S$ contains 16 distinct lines unless it is not smooth, in the generic singular case (one double point) four of its lines count twice and pass through the double point $t$ (see [Dm]). The quadrics in $l_{pq}$ all have the same Zariski tangent space at $t$ so at least one of them is singular at $t$; as $Q$ parametrizes the singular quadrics this has to be one of the intersections points of $Q$ with $l_{pq}$ say $q_1$. Taking $(\pi p, \pi q)$ to be one of the lines through $t$:

$$h^0(g_1 - \pi p - \pi q) > 0 \quad \text{and} \quad h^0(K_X - g_1 - \pi p - \pi q) > 0$$

Then $l_{pq}$ is tangent to $Q$ at $q_1$ and if $Y$ is tetragonally related to $X$ via $g_1$, $\Theta \cdot \Theta_x$ contains Prym-embeddings of $\tilde{Y}$ and $\tilde{Y}_\lambda$; as in i) $H$ is tangent to $T$.

Case iii) corresponds to the second component of the branch locus.

Generically on this component we see that $\tilde{h}^{-1}(H)$ has $2^7 - 2$ elements in $\tilde{A} \setminus E$ and one element in $E$ that counts with multiplicity two. Q.E.D.

(5.26) Remark: Notice that in case i) above $l_X$ is one of the lines in $T_H = T \cap H$ passing through the singular point and that in case ii) $l_X$ does not pass through the singular point of $T_H$.

Next we wish to prove the assertions of 1.6 about the inverse image of $T^*$ in $A$ and to give a characterization of $B$.

Looking back at the proof of 5.25 and at 5.26, we see that, taking $l_X$ to be one of the lines which do not contain the double point of $T_H$, $h^{-1}(H) \cap \Sigma(X)$ has (generically)
48 distinct elements, 16 of them correspond to the four lines in $S$ which pass through the
double point $t$ of $S$ and in fact the 32 others are not ramification points of $h$, but if $x \in R'$ is one of these points, for each line $l_Y$ through the double point of $T_H$, $\Theta, \Theta_x$ contains only one of the Prym-curves corresponding to it: $\Theta, \Theta_x$ contains 21
Prym-embedded curves, six of them counting twice. So we conclude:

(5.27) **PROPOSITION**: If an intersection $\Theta, \Theta_x$ contains a Prym-embedded curve $\tilde{X}$
and $\tilde{X}_\lambda$, then it is the Fano variety of lines of a double solid.

We wish to say a few words about points of order 2. We first observe

(5.28) **LEMMA**: The map $1: P^{-1}(A) \to F$, which to each Prym-curve $X$ associates its
line $l_X$, is a double cover.

**Proof**: As $A$ is generic, every $\Gamma_{00}$-divisor is irreducible. So the base locus of the pencil
$l_X$ is always a surface. (It is equal to $\Sigma(X) \cup \Sigma(X_\lambda)$ for $X$ smooth.) So $l$ is a
morphism. Also $l$ does not contract any curve. As the fano surface of lines $F$ of $T$ is
smooth we have that $l$ is also a finite morphism. Hence $l$ is a double cover. Q.E.D.

(5.29) Fix a curve $X \in P^{-1}(A)$, let $Q \subset N$ be as before. As $l_X = l_{X_\lambda}$ (5.1) we deduce
that $N$ can also be identified with the net of quadrics containing $\kappa X_\lambda$, and $Q$ is also the
plane quintic parametrizing singular quadrics containing $\kappa X_\lambda$. Therefore, on $JQ$ we are
given three points of order 2:

- $\alpha$ with associated double cover $\tilde{Q} \to \tilde{Q}$ and $P(Q, \alpha) = JT$
- $\alpha'$ with associated double cover $\text{Sing}\Theta' \to Q$ and $P(Q, \alpha') = JX$
- $\alpha''$ with associated double cover $\text{Sing}\Theta'' \to Q$ and $P(Q, \alpha'') = JX_\lambda$.

Let $\hat{Q}$ be the inverse image of $\tilde{Q}$ in $P^{-1}(A)$, then, by 3.1, $\hat{Q}$ is a fiber product
in two (distinct) ways:

\[
\begin{array}{ccc}
\hat{Q} & \to & \tilde{Q} \\
\downarrow & & \downarrow \\
\text{Sing}\Theta'' & \to & Q \\
\end{array}
\]

\[
\begin{array}{ccc}
\hat{Q} & \to & \tilde{Q} \\
\downarrow & & \downarrow \\
\text{Sing}\Theta' & \to & Q \\
\end{array}
\]
The involution of $\tilde{Q}$ for the cover $\tilde{Q} \to \tilde{Q}$ is just $\lambda$. Let $\sigma_{\alpha}$ and $\sigma_{\alpha}''$ be the involutions of $\tilde{Q}$ for the covers $\tilde{Q} \to \tilde{Q}' = \text{Sing}\Theta'$ and $\tilde{Q} \to \tilde{Q}'' = \text{Sing}\Theta''$. As generically $X \neq X_{\lambda}$, $\sigma_{\alpha} \neq \sigma_{\alpha}''$.

Let $q$ be an element of $Q$. Let $g_1$ and $h_1$ be the two opposite $g_4^1$'s corresponding to the two elements of $\text{Sing}\Theta' \subset JX$ above $q$ (see 1.3). Let $Y$ and $U$ be tetragonally related to $X$ via $g_1$, then (3.1) $Y_{\lambda}$ and $U_{\lambda}$ are tetragonally related to $X$ via $h_1$, i.e., $\sigma_{\alpha}(Y) = U$ and $\sigma_{\alpha}(Y_{\lambda}) = U_{\lambda}$. Similarly $\sigma_{\alpha}''(Y_{\lambda}) = U_{\lambda}$.

Therefore, the three involutions commute and the product of two of them is equal to the third one.

The above involutions are the nonzero elements of a group of automorphisms of $\tilde{Q}$ isomorphic to $(\mathbb{F}_2)^2$.

Hence the points $\alpha, \alpha', \alpha''$ commute and are the nonzero elements of a vector-subspace $V^2$ of $(JQ)_2$.

The quadratic form on $(JQ)_2$ is given by $h^0(g_5^2 \otimes \cdot)$ modulo 2. The antisymmetric bilinear form on $(JQ)_2$ is given by $(M2)$

$$(\beta, \beta') \equiv h^0(g_5^2) + h^0(g_5^2 \otimes \beta) + h^0(g_5^2 \otimes \beta') + h^0(g_5^2 \otimes \beta \otimes \beta') \mod 2$$

By a straightforward computation we deduce that $V^2$ is totally isotropic with respect to $(\cdot, \cdot)$.

Therefore in the exact sequences

$$0 \to \{\alpha\} \to \{\alpha\}^\perp \to (JT)_2 \to 0$$
$$0 \to \{\alpha'\} \to \{\alpha'\}^\perp \to (JX)_2 \to 0$$
$$0 \to \{\alpha''\} \to \{\alpha''\}^\perp \to (JX_{\lambda})_2 \to 0$$

associated to our Prym constructions ([M3]) $V^2$ projects to the three points $\mu$, $\eta$ and $\eta_{\lambda}$. The Prym variety of the last two is $A$. The first one gives by restriction the double cover $1 : P^1(A) \to F$ where $F$ is the variety of lines in $T$ (by [CG] the albanese variety
of $F$ is isomorphic to $JT$). A consequence of the above discussion is

The point $\mu$ of order 2 in $JT$ determined by $A$ is even.

Here "$\mu$ even" is equivalent to the fact that $\mu$ is the image of even points of order 2 on $JQ$.

(5.30) Next we mention some geometric "pictures" that summarize the relations given by $\lambda$ and the tetragonal construction.

Consider a plane section of $T$, union of three lines $l_X, l_Y, l_U$. By 5.12 we can suppose that $\{X,Y,U\}$ is a tetragonal related triple. The three plane quintics $Q, Q_1, Q_2$ associated to these lines with their double covers $\bar{Q}, \bar{Q}_1, \bar{Q}_2$ form a tetragonal triple:

Let $t \in Q$ be the common image of $l_Y = l(Q_1)$ and $l_U = l(Q_2)$ in $N$. Each line incident to $l_Y$ picks via incidence a lifting in $\bar{Q}$ of a divisor of $g_5^2 - t$; similarly for lines incident to $l_U$.

The curves $W'$ and $W''$ tetragonally related to $(\text{Sing}\Theta', Q)$ through $g_5^2 - t$ are trigonal because the common Prym variety of the tetragonal triple is $JX$: by [B1] the only smooth curves with Prym variety a jacobian of dimension 5 are trigonal curves and plane quintics (there is only one plane quintic with Prym $JX$). The same is true for the tetragonal triple obtained from $(\text{Sing}\Theta'', Q)$, and also for those obtained with $Q_1$ and $Q_2$.

Donagi observes that the curves in every tetragonally related triple are trigonally related to a curve $W$ (here of genus 7). The curve $W$ comes with the three points $\beta, \beta_1, \beta_2$ of order 2 with respective Prym varieties $JQ, JQ_1, JQ_2$ and with a totally isotropic $\mathbb{F}_2$-vector space $V^3$ of dimension 3 which is the common inverse image of the totally isotropic vector spaces in $(JQ)_2$ (called $V^2$ in 5.29), $(JQ_1)_2, (JQ_2)_2$ by the exact sequences analogous to those in 5.29.

So we have seven curves of genus 6 whose jacobians are Pryms of this trigonal curve for the elements of $\mathbb{P}V^3$. Three of these are the plane quintics and the other 4 are
the trigonals \( W', W'' \) and their analogues. The three points \( \beta, \beta_1, \beta_2 \) are on a line in \( \mathbb{P}V^3 \). The same is true for every set of three points associated to a tetragonal triple. Hence each line is associated to an abelian variety of dimension 5 which is the common Prym variety of the curves on that line; six of these are Jacobians and the last one is \( JT \); they all come with a point of order 2 image of \( V^3 \) (these are \( \mu \) and \( \eta \)'s with our previous notations). The above was done by Donagi in reverse order (start with a trigonal \( W \) and \( V^3 \)): this is how he introduced \( (T, \mu) \) for a generic ppav.

The space \( \mathbb{P}V^3 \) is a triangle.

The common Prym variety for the abelian varieties of dimension 5 in the configuration (in the more general sense of Mumford in [M3] "Prym varieties" can be defined for every abelian variety with certain data) is \( A \).

The space \( V^3 \) is a cube. One of the vertices of the cube is the origin.

Now dualize the cube: we obtain another cube with vertices: the origin, \( T, X, Y, U, X_\lambda, Y_\lambda, U_\lambda \).

The projectivization of the cube is a triangle: \( \mathbb{P}(V^3)^* \).

Embed \( \mathbb{P}(V^3)^* \) in \( \mathbb{P}^5(\mathbb{F}_2) \) by the Veronese map. Then project from \( T \): we obtain an octahedron with vertices the six straight lines in the triangle or the six planes in the cube that are also planes in the euclidian sense. Each vertex corresponds to one of the curves \( X, Y, U, X_\lambda, Y_\lambda, U_\lambda \). Painting in white a face of the octahedron whose vertices are tetragonally related curves and painting the other faces alternatively in black and white, all white triangles correspond to tetragonally related triples and black triangles correspond to non-tetragonally related triples.

See Figures 1 - 5 for the Triangle, the Cube, the dual of the Cube, the dual of the Triangle and the Octahedron respectively.
Figure 1: The Triangle

Figure 2: The Cube
Figure 3: The Dual of the Cube

Figure 4: The Dual of the Triangle
Figure 5: The Octahedron
6. RELATION WITH PPAV'S OF DIMENSION 5

We first need some preliminaries on Prym-embeddings of curves in the theta divisor - denoted by $\tilde{\Theta}$ - of a five-dimensional abelian variety $B$.

Let $Y$ and $U$ be two tetragonally related curves with $B$ as common Prym variety. Let $g^1_4 \in W^1_4(U)$ relate $U$ to $Y$ and $g^2_6 = K_U - g^1_4$. Suppose that $B$ is not a hyperelliptic jacobian and that, if $U$ is bielliptic, $g^2_6$ is not the pullback of a $g^2_3$ on an elliptic curve.

(6.1) Lemma: The family $\tilde{Y}$ of Prym-embeddings of $\tilde{Y}$ in $\tilde{\Theta}$ is a special subvariety associated to $g^2_6$ ([B3]), i.e., one of the two components of

$$\{E \in \text{Pic}^6\tilde{U} : h^0(E) > 0, \pi_*E \equiv g^2_6\}$$

so by [BD1] its homology class is $3[\Theta]^3/3! = [\Theta]^3/3$.

Proof: Prym-embeddings of $\tilde{Y}$ are translates of one component of

$$\{E \in \tilde{\text{U}}^{(4)} : \pi_*E \equiv g^1_4\}$$

so they are given by divisors $G \in \text{Pic}^6\tilde{U}$ such that $E + G \in \Theta$ for all $E \in \tilde{Y}$. This is equivalent to $\pi_*G \equiv K_Z - \pi_*E \equiv g^2_6$ and $h^0(E+G)$ is even and positive. This will be true for $G$ in one of the two components of $\{E \in \text{Pic}^6\tilde{U} : h^0(E) > 0, \pi_*E \equiv g^2_6\}$.

The rest of the proof is totally similar to the proof of 2.4.

Q.E.D.

Let $A \in \mathcal{C}_4$. By [M1] (page 227) the space of extensions of $A$ by $G_m$ is isomorphic to $A$ via the composition

$$A \rightarrow \text{Pic}^0A \rightarrow \text{Ext}^1(A,G_m)$$

where the first map is $x \mapsto \mathcal{O}_A(\Theta - \Theta_x)$ and the second one is $L \mapsto L \setminus o(A)$ where $o$ is the 0-section of the bundle $L$. 
For each $x \in A$, letting $A_x$ be the extension of $A$ with extension data $x$, we have a rational section $s : A \to A_x$ obtained from the rational section of $L(\Theta - \Theta_x)$ which is 0 on $\Theta$ and $\infty$ on $\Theta_x$. Composing $s$ with $\text{id} : A \to A$ we deduce that we have an isomorphism of extensions:

$$0 \to G_m \to A_x \to A \to 0$$

This permits us to construct the $\mathbb{P}^1$-bundle $\tilde{A}_x$ over $A$ by gluing the bundles $L(\Theta - \Theta_x)$ and $L(\Theta - \Theta_x)$ along $A_x \cong A_{-x}$. We obtain $\tilde{A}_x$ from $\tilde{A}_x$ by gluing the $0$-section $A_0 \cong A$ and the $\infty$-section $A_{\infty} \cong A$ (in terms of a parameter which is 0 on the zero section of $L(\Theta - \Theta_x)$ for instance) identifying $a \in A \cong A_0$ with $a + x \in A \cong A_{\infty}$ ([M4]).

By [BC] the image of $s$ in $\tilde{A}_x$ is the limit of the theta divisors of a generic family of five-dimensional ppav's with central fiber $\tilde{A}_x$. We will denote $\tilde{\Theta}^x = s(A)$ (resp. $\tilde{\Theta}^x$, resp. $\Theta^x$) in $\tilde{A}_x$ (resp. $\tilde{A}_x$, resp. $A_x$).

Let $X \in \mathbb{P}^1(A)$ be smooth such that $x = [p, q] \in \Sigma(X)$. Let $X_{pq} = X_{p=q}$ and \( \tilde{X}_{pq} = \tilde{X}_{p=q} \) \( \tilde{X}_{pq} = \tilde{X}_{p=q} \). We have (6.2) LEMMA: (i) There is a two-dimensional family of embeddings of $\tilde{X}_{pq}$ in $\tilde{\Theta}^x$ parametrized by the set

$$\{ E \in \text{Pic}^h \tilde{X} ; \pi^* E \cong K_X , h^0(E) = 1 , h^0(E - \x) = 1 , h^0(E - \sigma p - \sigma q) > 0 \}$$

These embeddings project to Prym-embeddings of $\tilde{X}$ in $A$.

(ii) The one-dimensional families of Prym-embeddings of $\tilde{X}$ in $\Theta$ and their inverse images by $t_x$ in $\Theta_x$ (which by intersection give us the 2 Prym-embeddings of $\tilde{X}_\lambda$ in $\Theta.x$) have the same image by $s$ in $\tilde{A}_x$ and give us a double curve in the boundary of the family in (i).

Proof: Parametrize $A$ with $X$ as in 2.4, then all Prym-embeddings of $\tilde{X}$ in $A$ are of
the form
\[ j_E : \tilde{X} \to A \]
\[ p \mapsto p - \sigma p + E \]
where \( E \in \text{Pic}^8\tilde{X} \) is such that \( \pi_E E = K_X \) and \( h^0(E) \) is odd. As before \( j_E(\tilde{X}) = \tilde{X}_E \subset \Theta \) (resp. \( \Theta_x \)) if and only if \( h^0(E) = 3 \) (resp. \( h^0(E+x) = 3 \)). (\( h^0(E) \geq 5 \) is excluded by Clifford's lemma)

So the first conditions on \( E \) are \( h^0(E) = 1 \) and \( h^0(E - x) = 1 \) because \( s \) sends \( \Theta \) and \( \Theta_x \) into \( A_0 \) and \( A_\infty \) in a 1-to-1 way.

Now
\[ j_E^*\Theta = \{ p ; h^0(p - \sigma p + E) > 0 \} \]
\[ j_E^*\Theta_x = \{ p ; h^0(p - \sigma p + E - x) > 0 \} \]

So we get
\[ j_E^*\Theta = \{ p ; h^0(p + E) > 1 \text{ or } h^0(E - \sigma p) > 0 \} \] and as \( \sigma E + E = K_X \)
\[ j_E^*\Theta = \{ p ; h^0(E - \sigma p) > 0 \} = \sigma E \]

similarly
\[ j_E^*(\Theta_x - \Theta) = x \]

So we obtain in \( \overline{\tilde{A}_x} \) a copy of \( \tilde{X} \) with 8 points identified unless some of the points are in \( \Theta,\Theta_x \) (because \( \Theta,\Theta_x \) is blown up by \( s \)), i.e., equal. We need
\[ \sigma E = \sum_{1 \leq i \leq 6} p_i + p_7 + p_8 \]
\[ \sigma E - \sigma x = \sum_{1 \leq i \leq 6} p_i + q_7 + q_8 \]
so \( q_7 + q_8 - p_7 - p_8 = x \) that is \( p_7 + p_8 = \sigma(p + q) \), \( q_7 + q_8 = p + q \).

The rest of the assertions are now clear.

Q.E.D.

(6.3) **Corollary**: \( A_x \) is the Prym variety of the cover \( \tilde{X}_{pq} \to X_{pq} \).

**Proof**: By lemma 6.2 we have a surjective albanese map \( \text{alb} : J\tilde{X}_{pq} \to A_x \) with a commutative diagram:

\[
\begin{array}{ccc}
\text{alb} : J\tilde{X}_{pq} & \to & A_x \\
& & \downarrow \\
& & JX
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \\
J\tilde{X} & \to & JX
\end{array}
\]
So the composition of the top horizontal arrows vanishes on pullbacks of divisor classes on $X_{pq}$, which means that $\text{alb}(\pi*JX_{pq}) \subset \mathfrak{C}^* = \text{Ker} (A_x \rightarrow A)$. So we have a group homomorphism

$$\mathfrak{C}^* \rightarrow JX_{pq} \rightarrow \mathfrak{C}^*$$

If it is nonzero, it has to be an isomorphism because it is induced by a linear homomorphism of $\mathfrak{C}$ via the exponential map:

$$T_1 \mathfrak{C}^* = \mathfrak{C} \rightarrow \mathfrak{C} = T_1 \mathfrak{C}^*$$

$$\exp. \downarrow \quad \downarrow \exp.$$  

$$\mathfrak{C}^* \rightarrow \mathfrak{C}^*$$

Then the sequence

$$0 \rightarrow \mathfrak{C}^* \rightarrow JX_{pq} \rightarrow JX \rightarrow 0$$

would split ($x$ is generic). So our albanese map factors

$$\text{alb}' : J\tilde{X}_{pq}/JX_{pq} \rightarrow A_x$$

The kernel of $\text{alb}'$ is contained in

$$\mathfrak{C}^* = (\mathfrak{C}^*)^2/\mathfrak{C}^* = \text{Ker} (J\tilde{X}_{pq} \rightarrow J\tilde{X}) / \text{Ker} (JX_{pq} \rightarrow JX)$$

and it is finite. However under $\text{alb}'$, $\mathfrak{C}^*$ goes to $\mathfrak{C}^*$. As before the restriction of $\text{alb}'$ to $\mathfrak{C}^*$ has to be an isomorphism.

Hence the family in 6.2 is the family of Prym-embeddings of $\tilde{X}_{pq}$ in $\overline{Θ^x}$. We have

(6.4) **COROLLARY**: Generically on $\mathfrak{C}xt\mathfrak{Q}_4$, the family of Prym-embeddings of $\tilde{X}_{pq}$ in $\overline{Θ^x}$ is reduced.

**Proof**: We need to consider the compactifications of the families in 6.2 that are the limits of the generic cycles. By 6.2 the underlying variety of our cycle for generic $A_x$ is the image under $s$ of

$$\{E \in \text{Pic}^0\tilde{X} ; \pi_*E \equiv K_X , h^0(E) = 1 , h^0(E + x) = 1 , h^0(E - \sigma p - \sigma q) > 0\}$$

and the compactification is the image under $s$ (in $\overline{A_x}$) of
\{ E \in \text{Pic}^8 \bar{X} ; \pi_* E = K_X , h^0(E) \text{ odd} , h^0(E - \sigma_p - \sigma_q) > 0 \}

By [B3] this has the same homology class as
\{ E \in \text{Pic}^8 \bar{X} ; \pi_* E = K_X , h^0(E) \text{ even} , h^0(E - \sigma_p - \sigma_q) > 0 \}

and by [BD1] this is equal to \( \Theta . \Theta_{[p,q]} \), so has homology class \( [\Theta]^2 \) in \( A \). Now with the notations of [BC]:

\[ [\Theta]^2 = \sum_{1 \leq i < j \leq 4} \gamma_i x \delta_i x \gamma_j x \delta_j \]

and the homology class of the image in \( \bar{A}_x \) is
\[ 3 . [\Theta]_3^2 = [\Theta]_3^2 . \]

Q.E.D.

Together with the above, the following gives a second proof of the result of [DS], namely, that the degree of \( \mathcal{P} \) from \( \mathcal{P}_6 \) to \( \mathcal{Q}_5 \) is 27.

(6.5) PROPOSITION: The map \( \mathcal{P} \) is generically unramified on \( \mathcal{Q}_5 \).

Proof: The space \( H^0(X_{pq}, \omega_{pq} \otimes \Omega_{pq}) \) is the cotangent space to the deformations of \( X_{pq} \) ([L] and [Sc]).

The codifferential of \( \mathcal{P} \) is multiplication:

\[ S^2 H^0(X_{pq}, \omega_{pq} \otimes \eta_{pq}) \rightarrow H^0(X_{pq}, \omega_{pq} \otimes \Omega_{pq}) \]

where \( \omega_{pq} \) is the dualizing sheaf of \( X_{pq} \), \( \Omega_{pq} \) is the sheaf of regular differentials on \( X_{pq} \) and \( \eta_{pq} \) is the point of order 2 associated to the cover \( \bar{X}_{pq} \rightarrow X_{pq} \).

If the Prym map is ramified at \( (\bar{X}_{pq} - X_{pq}) \), then there is a quadric \( q \) in \( \mathbb{P}^4 = \mathbb{P}T_0A_\chi \) containing the Prym-canonical image \( \chi_{X_{pq}} \) of \( X_{pq} \). Let \( o \) be \( \mathbb{P}T_0A^* \subset \mathbb{P}^4 \).

Then \( o \) is the singular point of \( \chi_{X_{pq}} \) and \( o \in q \).

(6.6) Claim: The quadric \( q \) is singular at \( o \).

Suppose that this is not the case and let \( T \equiv \mathbb{P}^3 \) be \( \mathbb{P}T_0q \). Let

\[ v : \mathbb{P}^4 = \mathbb{P}T_0A_\chi \rightarrow \mathbb{P}T_0A = \mathbb{P}^3 \]

be the projection from \( o \). Consider the cone \( \mathcal{E}X \) over the Prym-canonical image \( \chi_X \) of \( X \): this is a surface of degree 8 in \( \mathbb{P}^4 \), its intersection with \( q \) is a curve of degree 16 which contains \( \chi_{X_{pq}} \). Say \( \mathcal{E}X \cap q = \chi_{X_{pq}} + S \).

As \( o \) is the vertex of \( \mathcal{E}X \), \( \mathcal{E}X \) has a point of multiplicity 8 at \( o \) and \( S \) has a
point of multiplicity 6 at \( o \). However \( S \) is of degree 6 so its projection in \( \mathbb{P}^3 \) is a finite union of points: \( S = \sum_{1 \leq i \leq 6} l_i \) where \( l_i \) is a line through \( o \). Let \( p_i = v(l_i) \), then as \( T \) is a projective space of dimension 3 we have:

\[
\sum_{1 \leq i \leq 6} p_i + \pi p + \pi q = K_X + \mu_X
\]

so by the genericity assumption on \( p \) and \( q \) we can suppose that we have at least 5 distinct \( p_i \)'s (or \( l_i \)'s).

Choose a generic quadric in \( \mathbb{P}^3 \) containing the \( p_i \)'s and let \( q' \) be the cone over it in \( \mathbb{P}^4 \). The intersection \( q \cap q' \) is a quartic surface containing the \( l_i \)'s and also \( T \cap q \cap q' \) contains the \( l_i \)'s so, as we have at least five distinct \( l_i \)'s

\[
q \cap q' \subset T, \quad v(q') = v(q \cap q') = v(T) \text{ is a plane.}
\]

However \( v(q') \) is a generic quadric containing the \( p_i \)'s and it has rank greater than 2.

Now, \( q \) being singular at \( o \), \( v(q) \) is a quadric in \( \mathbb{P}^3 \) containing \( kX \), this means that \( P : \mathbb{P}^5 \to \mathcal{Q}_4 \) is ramified at \( X \), but \( A \) is generic and this can not be.

Q.E.D.

Next we would like to determine the ramification locus of \( P \) (there is an unpublished proof of this in [Do3]).

By the analysis of \( h \) in section 5 (see especially the discussion preceding 5.27) we see that the number of curves in \( \Theta.\Theta_x \) (hence the cardinality of the fiber of \( P \) at \( A_x \)) drops exactly when \( x \in R' \). So we are looking for extensions \( A_x \) which contain six Prym curves whose corresponding lines intersect in one point on \( T \). The first place to look for them is among generalized intermediate jacobians of double solids.

Let \( x \) be generic in \( R' \) and \( z \) be the point on \( T \) corresponding to \( h(x) \). Let \( (X_1)_x, \ldots, (X_6)_x \) be the curves in \( \Theta.\Theta_x \) which count with multiplicity 2. Then if \( x = [s_i, t_i] \in \Sigma(X_i) \), we have Prym-embeddings of \( (X_i)_{s_i, t_i} \) in \( A_x \).

Let \( Z \) be a double solid above \( z \) with double points \( p_1, \ldots, p_6 \) such that the set
of lines in $Z$ through $p_1$ is $(\tilde{X}_1)_\lambda$ and the sets of lines in $Z$ through $p_2, \ldots, p_6$ are respectively $\tilde{X}_2, \ldots, \tilde{X}_6$. Let $\tilde{Z}$ be the blow up of $Z$ at $p_2, \ldots, p_6$.

(6.7) Lemma: The extension data for the generalized intermediate jacobian $J\tilde{Z}$ of $\tilde{Z}$ is $\pm x$.

Proof: First notice that there is no Prym-embedding of $(\tilde{X}_1)_\lambda$ (with two pairs of points identified) in $J\tilde{Z}$, because such an embedding would have to be the image of $(\tilde{X}_1)_\lambda \subset E\tilde{Z}$ by $A\psi$, but this curve has two singular points of multiplicity 6 (because we do not blow up $p_1$ in $\tilde{Z}$). Letting $y$ be the extension data for $J\tilde{Z}$ we deduce that $y \in R'$.

Actually $y = \pm x$:

$\Theta, \Theta_x$ and $\Theta, \Theta_y$ contain the same Prym-curves and

by letting the Cremona group $\{i_\mathcal{G}, \mathcal{G} \subset \{p_1, \ldots, p_6\}\}$ act (see 1.2 and the proof of 3.5) we obtain 32 different sets of 6 curves with associated lines $l_{X_1}, \ldots, l_{X_6}$ so 64 different values for $y$ (counting the negatives of the extension data's).

Q.E.D.

By [B1] the generalized Prym mapping (still denoted by $P$) is proper on $\mathcal{C}_5$. Also $\text{Pic}\mathcal{C}_5 \cong Z$ by [SV]. Hence the divisor classes of all the components of the branch locus of $P$ are ample. We infer that their closures in $\mathcal{C}_5 \cup \text{Ext}\mathcal{C}_4$ must meet $\text{Ext}\mathcal{C}_4$.

Also the branching of $P$ on $\text{Ext}\mathcal{C}_4$ is simple along the locus of generalized jacobians of double solids because each ramification point counts twice and not more (generically). Thus the branch locus is irreducible and it can only be the closure of the locus of intermediate jacobians of double solids with five ordinary double points.
7. SCHOTTKY

In this section our aim is to prove what was announced in 1.7, namely, the main result will be that the base locus \( V(\Gamma_{00}) \) of \( |2\Theta_{00}| \) is reduced to 0 if \( A \) is outside the locus \( \overline{\mathcal{J}}_4 \cup \mathcal{Q}_{n11} \). Notice that, for \( A \) generic, we proved this in proposition 5.15. We define \( \mathcal{Q}_{n11} \) to be the locus of Prym varieties of \( (\tilde{X},X) \) where

- either \( X \) is the union of two elliptic curves meeting in four points, (these ppav's are isogenous but not isomorphic to a product of two ppav's of dimension 2)
- or \( X \) is irreducible with one double point and its normalization has a vanishing theta-null. Furthermore, in this case we suppose that \( A \) has a two-dimensional family of such Prym-curves (in the first case this is automatically verified).

The locus \( \mathcal{Q}_{n11} \) has dimension 6.

Recall that \( \overline{\mathcal{J}}_4 \) is the closure, in \( \mathcal{Q}_4 \), of the locus of jacobians. In the following we will be using results of [B1] without mentioning it each time.

(7.1) THEOREM: If \( A \not\in \overline{\mathcal{J}}_4 \cup \mathcal{Q}_{n11} \), then the generic curve in \( P^{-1}(A) \) is smooth.

Proof: As products of ppav's of lower dimension are in the closure of \( \mathcal{J}_4 \), \( A \) is the (generalized) Prym variety of connected double covers \( \tilde{X} \) of curves \( X \) of arithmetic genus 5 such that

(i) \( X \) is smooth or has at most nodes as singularities
(ii) the double cover \( \tilde{X} \) is ramified exactly at the singular points of \( X \)
(iii) at the singular points of \( \tilde{X} \) the two branches are not exchanged under the covering involution \( \sigma \).

Then irreducible components of \( \tilde{X} \) correspond to irreducible components of \( X \).
Let $\tilde{X}_n$ and $X_n$ be the normalizations of $\tilde{X}$ and $X$ respectively. Let $\Omega_X$ be the sheaf of Kähler differentials on $X$. We have the exact sequence

$$0 \to \Theta_i \mathbb{C} \to \Omega_X \to \omega_X \to \Theta_i \mathbb{C} \to 0$$

where $\omega_X$ is the dualizing sheaf of $X$; the skyscraper sheaves have support on the singular locus of $X$. For each $p_i', p_i'' \in X_n$, projecting to a double point $p_i$ of $X$, the sheaf on the right is generated by $(1/t_i', -1/t_i'')$ for some local coordinates $t_i'$ and $t_i''$ at $p_i'$ and $p_i''$. The sheaf on the left is generated by $t_i'dt_i''$ and $-t_i'dt_i''$. The tangent space at $X$ to the space of deformations of $X$ is $\text{Ext}^1(\Omega_X, \Theta_X)$ (global Ext); this has dimension 12.

We have an exact sequence

$$0 \to H^1(X, \text{Hom}(\Omega_X, \Theta_X)) \to \text{Ext}^1(\Omega_X, \Theta_X) \to H^0(X, \text{Ext}^1(\Omega_X, \Theta_X)) \to 0$$

where $\text{Ext}^1$ is the local Ext; a skyscraper sheaf supported on the singular locus of $X$ (see [L] and [Sc]). The space $H^1(X, \text{Hom}(\Omega_X, \Theta_X))$ is the tangent space at $X$ to the space of deformations of $X$ that are at least as singular as $X$, if $s$ is the number of singular points of $X$, the dimension of this space is $12 - s$. Using Serre Duality, we see that the cotangent space to the space of deformations of $X$ is $H^0(X, \Omega_X \otimes \omega_X)$.

Tensor the sequence (B) with $\omega_X$, we obtain a cohomology sequence:

$$0 \to \Theta_i \mathbb{C} \to H^0(X, \Omega_X \otimes \omega_X) \to H^0(X, \omega_X \otimes \omega_X)$$

Let $T^*$ be the image of $H^0(X, \Omega_X \otimes \omega_X)$ in $H^0(X, \omega_X \otimes \omega_X)$). Then the exact sequence

$$0 \to \Theta_i \mathbb{C} \to H^0(X, \Omega_X \otimes \omega_X) \to T^* \to 0$$

is the dual of the sequence (C) and $T = H^1(X, \text{Hom}(\Omega_X, \Theta_X))$. The skyscraper sheaf is the vector space of sections of $\Omega_X \otimes \omega_X$ with support on the singular locus of $X$.

In the canonical space of $\tilde{X}$ the double points of $\tilde{X}$ are in the $(-1)$-eigenspace of the involution of $[K_X]^*$ induced by $\sigma$ (because they are fixed by $\sigma$ and project to the double points of $X$, hence cannot be in the $(+1)$-eigenspace). Thus the Prym-canonical image $\chi_X$ of $X$ is the union of $\chi X_n$ (curve of degree $8 - s$ and genus $5 - s$) and the lines connecting $p_i'$ to $p_i''$. 
From this and also [B1] (p. 170) we see that we can replace $X$ by the union of $X_n$ and $\mathbb{P}^1$'s joining $p_i'$ to $p_i''$. This gives a flat family of curves over $\mathbb{P}_5$ and we have a morphism from this family to the bundle over $\mathcal{O}_d^*$ with fiber $\mathbb{P}T_0A$ at $A$.

For $X$ smooth, the codifferential of $P$ is multiplication

$$S^2H^0(X,\omega_X \otimes \eta) \to H^0(X,(\omega_X)^2)$$

as the codifferential of $P$ is a bundle map from the cotangent bundle of $\mathcal{O}_d^*$ to the cotangent bundle of $\mathbb{P}_5$, we deduce that it is multiplication everywhere. It also follows from the above considerations that the kernel of the codifferential of the restriction of $P$ to the space of curves that are at least as singular as $X$ is the space of quadrics in $T_0A$ which contain $\chi X_n$.

Consider the ramified cover $\pi_n : \tilde{X}_n \to X_n$. The variety $P(\tilde{X}_n, X_n)$ is isogenous to $P(\tilde{X}, X)$. Let $\delta$ be the divisor class on $X_n$ such that

$$\pi_n^* \mathcal{O}_{\tilde{X}_n} \cong \mathcal{O}_{X_n} \oplus \mathcal{O}_{X_n}(\delta)$$

and let $\Delta$ be the ramification divisor of $\pi_n$ ($\Delta \equiv 2\delta$), then the space of quadrics containing $\chi X_n$ is also the kernel of multiplication

$$S^2H^0(X,\omega_X \otimes \eta) = S^2H^0(X_n,\omega_{X_n}(\delta)) \to H^0(X_n,(\omega_{X_n})^2(\Delta))$$

(Notice that we also obtain an isomorphism

$$H^0(X_n,(\omega_{X_n})^2(\Delta))^* \cong H^1(X,\text{Hom}(\Omega_X, \mathcal{O}_X))$$

(7.2) First suppose that $A$ is simple, i.e., does not contain any abelian subvariety of smaller dimension, or equivalently, is not isogenous to a product of smaller dimensional abelian varieties ([M1] p. 173). Then every curve $X$ in $P^{-1}(A)$ is irreducible. Also, as $A$ is not in $\mathcal{O}_d$, $\tilde{X}$ is irreducible.

For $X$ singular we have the following possibilities:
<table>
<thead>
<tr>
<th>genus of $X_n$</th>
<th># of double points of $X$</th>
<th>genus of $X_n$</th>
<th>dimension of moduli of $(X,X)$</th>
<th>the generic $Q$ parameterizing singular quadrics containing $KX$</th>
<th>image by $P$</th>
<th>ppav's isogenous to a hyperelliptic jacobian by an isogeny of degree 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>8</td>
<td>$9+2 = 11$</td>
<td>1 double point (irreducible)</td>
<td>$Q_4^4$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>7</td>
<td>$6+4 = 10$</td>
<td>2 d. p. (irred.)</td>
<td>$Q_4$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>$3+6 = 9$</td>
<td>3 d. p. (irred.)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
<td>$1+8-1 = 8$</td>
<td>4 d. p. (irred.)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>4</td>
<td>$10-3 = 7$</td>
<td>5 d. p. (irred.)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We are going to show that in each case the family of singular curves of the given type in $P^1(A)$ is at most 1-dimensional.

Consider the restriction $P_s$ of $P$ to the space $\mathcal{F}_{6,s}$ of irreducible singular curves with exactly $s$ double points.

Let us first consider the case where $X$ has one double point. The moduli space of such curves is of codimension 1 in $\mathcal{F}_6$. We have to show that $\chi X_n$ is not contained in any quadric.

If $\chi X_n$ is contained in a nonsingular quadric $q$, write $X_n = aL_1 + bL_2$ in $q$, $L_1$ and $L_2$ being the two rulings of the quadric. We must then have $(a-1)(b-1) = \text{arithmetic genus of } \chi X_n$ and $a+b = 7 = \text{degree of } \chi X_n$. Hence up to a transposition of $a$ and $b$ we have two cases:

- $a = 2$ and $b = 5$. In this case $X_n$ is hyperelliptic. The ppav $A$ is the Prym of a double cover of $X_n$ ramified at two points: by [B1] (p. 171), $A$ is a jacobian.

- $a = 3$ and $b = 4$. This is a possible case. In this case the arithmetic genus of $\chi X_n$ is 6 and $\chi X_n$ has two double points. Let $\xi$ and $g_0$ be respectively the $g^1_3$ and the $g^1_4$ cut on $X_n$ by the rulings of $q$. Then
\[ K_{X_n}^{} + \delta = g_0^{} + \xi \]

Let \( \{u'; u''\} \) and \( \{v'; v''\} \) be the pairs of points of \( X_n \) that give us the double points of \( \chi X_n \). Then \( K_{X_n}^{} + \delta - u'' - v'' \) and \( K_{X_n}^{} + \delta - v'' - v'' \) move in two linear systems of dimension 2 and we can write

\[ K_{X_n}^{} + \delta - u'' - u'' = K_{X_n}^{} - t \quad \text{and} \quad K_{X_n}^{} + \delta - v'' - v'' = K_{X_n}^{} - t' \]

for some elements \( t \) and \( t' \) of \( X_n \). So

\[ \delta = u'' - u'' - t = v'' - v'' - t' \]

and

\[ \xi = K_{X_n} - g_0^{} + u'' - u'' - t = K_{X_n} - g_0^{} + v'' - v'' - t' \]

As \( \xi \) is a \( g_3^1 \), from this we first deduce that \( K_{X_n} - g_0 = t + t \) and then that

\[ \xi = u'' + u'' + t = v'' + v'' + t \]

Let \( p' \) and \( p'' \) be the points of \( X_n \) that we identify to obtain \( X \). Write \( \xi' = K_{X_n} - \xi \), then

\[ K_{X_n}^{} + \delta - p' - p'' = K_{X_n}^{} + \delta - 2 \delta = K_{X_n} - \delta = K_{X_n} - u'' - u'' - t = \xi' + \xi' - u'' - u'' + t = \xi' + t + t + t \]

So, in particular, \( K_{X_n} + \delta - p' - p'' - t' \) is a \( g_3^1 \) and the points \( p', p'', t, t' \) are on a line in \( \mathbb{P}T_0^4 \). So this line is in \( q \) and we have \( p' + p'' + t' + t \in g_0^{} \). It follows that

\[ \xi' = K_{X_n} + \delta - g_0 = K_{X_n} + \delta - p' - p'' - t' \quad \equiv \quad \xi' \]

So \( X_n \) is a curve of genus 4 with a vanishing theta-null. We also deduce that \( 2g_0 = K_X + \Delta \), so as \( h^0(g_0^{} - p' - p'') > 0 \), \( g_0 \) induces a vanishing theta-null on \( X \). Also notice that the pencil of planes through a quadriransient line to \( \chi X_n \) cuts, residually on \( X \), a singular vanishing theta-null. Moreover, \( h^0(X, \omega_X \otimes \eta(g_0)) = 2 = h^0(X, \eta(g_0)) \), so by [B1],

\[ A \in \Theta_{\text{null}} \]

The locus of the abelian varieties that have a two-dimensional family of these Prym-curves has dimension less than or equal to 6.

Suppose now that \( \chi X_n \) is contained in a singular quadric \( q \). We first notice that for degree reasons \( \chi X_n \) has to pass through the singular point of \( q \) and the ruling of \( q \) cuts a \( g_3^1 \) on \( q \). Suppose that we do have a two-dimensional family of curves with one
double point in $P^{-1}(A)$.

The abelian variety $J\tilde{X}_n$ is isogenous to the product $JX_n \times A$ by an isogeny of bounded degree. So $\tilde{X}_n$ and hence the ramification points of $\tilde{X}_n \to X_n$ are determined by $X_n$ and $A$. We deduce that the curves $X_n$ describe at least a curve in $\mathfrak{m}_4$. Equivalently, their jacobians describe at least a curve in $\mathfrak{g}_4$ (or $\mathfrak{g}_4'$). As $\text{Pic} \mathfrak{g}_4 = \mathbb{Z}$ ([SV]), the divisor $\theta_{\text{null}}$ is ample and intersects this (complete) curve or surface at (generalized) jacobians of curves $X'_n$ of arithmetic genus 4 with a vanishing theta-null.

Notice that for a generic $X_n$ the $g^1_3$ cut on $X_n$ by the ruling of $q$ verifies $2g^1_3 = K_{X'_n} + \delta - t$ where $\delta$ is the divisor class associated to the ramified cover $\tilde{X}_n \to X_n$ and $t$ is the vertex of $q$. The unique $g^1_3$ on a (fixed) $X'_n$ is the specialization of the (two) $g^1_3$'s on the generic $X_n$'s. So this is also true for the $g^1_3$ of $X'_n$. In this case we have in addition that $2g^1_3 = K_{X'_n}$, but this implies that $K_{X'_n} + \delta$ has a base point and $\chi X'_n$ has degree 6.

We deduce that the closure of $\mathcal{P}_{6,1}$ surjects onto $\mathfrak{g}_4$ with fibers of dimension 1 everywhere on $\mathcal{P}_{6,1}$.

If $s = 2$, for generic $X$, $\chi X_n$ is not contained in a quadric. Hence $P_2$ is generically étale and the closure of $\mathcal{P}_{6,2}$ surjects onto $\mathfrak{g}_4$. As $\chi X_n$ is of degree 6, there is at most one quadric containing it, so that the fibers of $P_2$ have dimension at most 1.

If $s = 3$, the dimension of $H^1(X, \text{Hom}(\Omega_{X}, \mathcal{O}_X))$ is 9. For all $X$, there is exactly one quadric containing $\chi X_n$. Hence $P_3$ is everywhere étale, in particular, the closure of its image is a divisor in $\mathfrak{g}_4$.

If $s = 4$, the dimension of $H^1(X, \text{Hom}(\Omega_{X}, \mathcal{O}_X))$ is 8. For all $X$, there is exactly one pencil of quadrics containing $\chi X_n$. Hence $P_4$ is also everywhere étale and the closure of its image has codimension 2 in $\mathfrak{g}_4$.

If $s = 5$, the dimension of $H^1(X, \text{Hom}(\Omega_{X}, \mathcal{O}_X))$ is 7. Then $\chi X_n$ is a twisted
and there is exactly a net of quadrics containing it. So $P_5$ is everywhere étale and the closure of its image has codimension 3 in $G_4$.

(7.3) Now suppose that $A$ is isogenous to a product of lower dimensional ppav’s. Then $P^{-1}(A)$ also contains reducible curves. The curve $X$ has at most four components. As in [B1] we associate a graph to $X$: the vertices of the graph are the irreducible components of $X$ and each edge between two vertices represents a point of intersection of the two irreducible components. Each irreducible component has to intersect the rest of the curve in at least four points because otherwise $A$ would be a product [B1]. We investigate the various possibilities for the graphs:

- We first see that there are at most two curves with four components in $P^{-1}(A)$:

In this case the genus of $X$ is greater than or equal to $12 - 3 = 9$

In this case $g \geq 11 - 3 = 8$

$g \geq 10 - 3 = 7$

$g \geq 9 - 3 = 6$

$g \geq 8 - 3 = 5$: this is a possible case but then all the components have to be of
genus 0, their double covers will be elliptic curves. Elliptic curves are rigid in an abelian variety hence there is only one curve of this form in $\mathbb{P}^1(\mathbb{A})$.

$g \geq 5$ : this case is analogous to the preceding case.

- Suppose now that $X$ has three components. Notice that we have to minimize the number of edges because they increase the genus. The possibilities are :

$g \geq 8 - 2 = 6$

$g \geq 7 - 2 = 5$ : $X$ has three rational components, $\tilde{X}$ has two elliptic components and one genus 2 component. The elliptic components embed in $A$ and do not move. The jacobian of the genus 2 component also embeds in $A$ and does not move either because it is an abelian subvariety of $A$. So there is only one such Prym-curve in $\mathbb{P}^1(\mathbb{A})$.

$g \geq 6 - 2 = 4$ : here we have two components $C_1, C_2$ of genus 0 and one component $C_0$ of genus 1. We are going to show that there is at most a one-dimensional family of such Prym-curves in $\mathbb{P}^1(\mathbb{A})$.

As each component $C_i$ intersects the rest of the curve in four points that are the ramification points of the cover $\tilde{C}_i \to C_i$, $\tilde{X}$ is the union of two elliptic curves and a curve of genus 3. The elliptic components embed in $A$ and are rigid in $A$. The Prym variety of the ramified cover $\tilde{C}_0 \to C_0$ embeds in $A$ and is the jacobian of a curve of genus 2. As the jacobian of the curve of genus 2 is an abelian subvariety of $A$, the curve of genus 2 does not move either. We claim that there is exactly a one-dimensional
family of such curves in \( P^1(A) \). For this it will be enough to show that there is exactly a one-dimensional family of ramified covers \( \tilde{C}_0 \to C_0 \) with Prym variety a fixed ppav of dimension 2. Let \( \delta \) be the divisor class on \( C_0 \) such that
\[
\tilde{C}_0 = \text{Spec}_{C_0} ( \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_0}(\delta) )
\]
then the Prym-canonical embedding of \( C_0 \) is given by \( \omega_{C_0}(\delta) \). This is of degree 2 and is onto from \( C_0 \) to \( \mathbb{P}^1 = \mathbb{P}T_0P(\tilde{C}_0, C_0) \). The codifferential of the Prym map from the space of curves of genus 1 with a cover ramified at four points to the space of curves of genus 2 is given by multiplication
\[
S^2H^0(C_0, \omega_{C_0}(\delta)) \to H^0(C_0, (\omega_{C_0}(\delta))^2)
\]
this is injective because the Prym-canonical image of \( C_0 \) is not contained in any quadric. Hence the fibers have dimension 1 everywhere as required.

- Suppose now that \( X \) has two components. We have the following possibilities:

\[ g \geq 6 - 1 = 5 : \text{all the components are rational. There is only one such Prym-curve in } P^1(A). \]

\[ g \geq 4 - 1 = 3. \text{ In this case either } X \text{ is the union of a rational curve and a curve of genus 2 or } X \text{ is the union of two elliptic curves.} \]

In the first case, \( \tilde{X} \) is the union of a curve of genus 5 and an elliptic curve meeting in four points. The fibers of the restriction of \( P \) to this locus have dimension 1: Arguing as in the last case where \( X \) has three components we have to show that, if \( C_0 \) is the component of genus 2 of \( X \), the map
\[
S^2H^0(C_0, \omega_{C_0}(\delta)) \to H^0(C_0, (\omega_{C_0}(\delta))^2)
\]
is injective. The image of \( C_0 \) in \( \mathbb{P}^2 = \mathbb{P}T_0P(\tilde{C}_0, C_0) \) is a quartic curve. If this quartic curve is twice a conic, then \( \omega_{C_0}(\delta) \) is equal to \( 3g_2^{1} \) and \( \delta \) is equal to the unique \( g_2^{1} \) of \( C_0 \). So the four points of ramification \( t_i \ (1 \leq i \leq 4) \) of \( \tilde{C}_0 \to C_0 \) verify (for instance) \( t_i \)
+ \tau_i \in \mathfrak{g}_2^1 \). The moduli space \( \mathcal{A}_1 \) of such curves \( X \) has dimension 6. Also, the moduli space \( \mathcal{A}_2 \) of curves corresponding to the last graph in the case of three components has dimension 6. It is immediately seen that the intersection \( \mathcal{A}_1 \cap \mathcal{A}_2 \) has dimension 5. We saw that the fibers of the restriction of \( P \) to \( \mathcal{A}_2 \) have dimension 1.

Suppose that at \( A \) the intersection \( P^{-1}(A) \cap \mathcal{A}_1 \) has dimension 2.

The jacobian of \( \tilde{C}_0 \) is isogenous to the product of \( C_0 \) and \( P(\tilde{C}_0,C_0) \) by an isogeny of fixed degree. So, if \( C_0 \) and \( P(\tilde{C}_0,C_0) \) are fixed, \( C_0 \) is fixed; hence the ramification points of \( \tilde{C}_0 \to C_0 \) are fixed. We deduce that the curves \( C_0 \) describe a family of dimension 2 in \( \mathcal{M}_2 \). Hence their Jacobians describe a divisor in \( \mathcal{A}_2 \) (or \( \mathcal{A}_2 \)). This must be ample because \( \text{Pic} \mathcal{A}_2 \cong \mathbb{Z}/10\mathbb{Z} \) (see [H]) so it encounters the boundary \( \mathcal{A}_2 \setminus \mathcal{A}_1 \). This implies that \( P^{-1}(A) \) intersects \( \mathcal{A}_2 \) and \( A \) is in \( P(\mathcal{A}_2) \subset \mathcal{A}_n \).

In the second case \( \tilde{X} \) is the union of two curves of genus 3 meeting in four points. Arguing as before we that the fibers of the restriction of \( P \) to this locus have dimension 2.

Q.E.D.

(7.4) Remark: It is proven in [B1] (p. 183) that a reducible curve of genus 5 with two elliptic components meeting in four points is a limit of smooth curves of genus 5 with a vanishing theta-null.

(7.5) Proposition: If \( A \) is not in \( \mathcal{A}_4 \cup \mathcal{A}_n \), then the only base point of \( 12\Theta_{10} \) is 0.

Proof: By 7.1, a generic curve in \( P^{-1}(A) \) is smooth (and irreducible).

The map \( \Sigma_A = (\cup \Sigma(X)) \to A \) is surjective for \( A \) generic, hence \( \Xi \to \mathfrak{K} \) or \( A \) is surjective (because the Prym map is proper and each \( \Sigma(X) \) is complete), hence \( \Sigma_A \to A \) is surjective for all \( A \).

As \( \tilde{h} \) is generically finite and \( \Sigma_A \to A \) is surjective, for \( X \) generic in a component of \( P^{-1}(A) \), the image of \( \Sigma(X) \) by \( \tilde{h} \) is two-dimensional. This is actually a plane because it is so for \( A \) generic. Hence there is exactly 1 pencil \( l_X \) of \( \Gamma_{00} \)-divisors
containing $\Sigma(X)$. The base locus of $l_X$ will be at most the union of $\Sigma(X)$ and a divisor which does not contain $\Sigma(X)$. Notice that, for homology class reasons, the map $X \rightarrow l_X$ is generically finite.

If the dimension of $V(\Gamma_{00})$ is $\geq 2$, then $\Sigma(X)$ will intersect the base locus because the homology class of $\Sigma(X)$ is $2[\Theta]^2$. Also $\Sigma(X)$ cannot be contained in the base locus by the surjectivity of $\Sigma_A \rightarrow A$.

By the corresponding statement for $A$ generic and because a generic curve in $P^1(A)$ is smooth, every element of $l_{2\Theta_{00}} \setminus l_X$ cuts on $\tilde{X}^{(2)}$ the inverse image of the set of pairs in $X^{(2)}$ which correspond to biseccants in a quadric containing $\kappa X$ (union the set of $(p, \sigma p)$). However the intersection of the quadrics containing $\kappa X$ does not contain any bisecant of $\kappa X$ unless $X$ is trigonal, in which case $P(\tilde{X}, X)$ is in the closure of $\mathcal{G}_4$.

Suppose now that $V(\Gamma_{00})$ has dimension less than 2.

Suppose that the base locus of every pencil $l_X$ (which we defined for $X$ generic in a component $F_0$ of $P^1(A)$) contains a divisor $D_0$. Then, as $\tilde{h}$ is generically finite, $D_0$ is the same for every pencil $l_X$. If the lines $l_X$ span a hyperplane, then every generic $\Gamma_{00}$-divisor intersects $D_0$ in the same surface: $V(\Gamma_{00})$ will be two-dimensional. So the lines $l_X$ are all in a plane: the pencil of hyperplanes containing this plane will be a line in $(l_{2\Theta_{00}})^* \subset$ contained in every plane $h(\Sigma(X))$. Looking at the inverse image of this line, we see that for every $X, Y \in F_0$, $\Sigma(X) \cap \Sigma(Y)$ is the same curve. Take $X$ and $Y$ tetragonally related and smooth and consider $\tilde{X}^{(2)}$ and $\tilde{Y}^{(2)}$ instead. Fix $X$ and let $Y$ vary. Then as above we can use the smoothness assumption and the generic result (5.11) to determine the inverse image of $\Sigma(X) \cap \Sigma(Y)$ in $\tilde{X}^{(2)}$. We conclude that as $X$ is nontrigonal, two of these cannot have common components (other than the set $\Delta' = \{(p, \sigma p) : p \in \tilde{X}\}$).

So the base locus of $l_X$ is two-dimensional. Notice that by 2.4, $X_\lambda$ is defined for smooth curves in $P^1(A)$. As $l_X$ is the limit of generic pencils $l_X$ and containment is
a closed condition we deduce that the base locus of \( l_X \) is be the union of \( \Sigma(X) \) and \( \Sigma(X_\lambda) \). Also, the \( X_\lambda \)'s describe a two-dimensional family because, as we saw in 2.4, if \( X \) is tetragonally related to \( Y \) through \( g_4^1 \) on \( Y \), then \( X_\lambda \) is tetragonally related to \( Y \) through \( K_X - g_4^1 \); hence if a one-dimensional family of \( X \)'s have the same \( X_\lambda \), then \( X_\lambda \) will have a two-dimensional family of \( g_4^1 \)'s and by \([B1]\) \( X_\lambda \) would be hyperelliptic. Hence a generic \( X_\lambda \) is smooth.

Any base point of \( |2\Theta|_{00} \) will be either in \( \Sigma(X) \) or in \( \Sigma(X_\lambda) \). This is taken care of as above. Q.E.D.

(7.6) COROLLARY: If \( A \) is not in \( \overline{\mathcal{D}}_4 \cup \mathcal{C}_{n11} \), then the base locus of \( |2\Theta|_{00} \) is \( 0 \) with multiplicity \( 2^8 \).

Proof: As the generic curve in \( P^1(A) \) is smooth, the proof of this is exactly like that of 5.15 and 5.22. Q.E.D.

As in 5.23 we have:

(7.7) COROLLARY: If \( A \) is not in \( \overline{\mathcal{D}}_4 \cup \mathcal{C}_{n11} \), the base locus of \( \tau|2\Theta|_{00} \) is empty.

(7.8) COROLLARY: If \( A \) is not in \( \overline{\mathcal{D}}_4 \cup \mathcal{C}_{n11} \), the linear system \( \tau|2\Theta|_{00} \) has projective dimension greater than or equal to \( 3 \). Equivalently, the Kernel of \( \tau: \Gamma_{00} \to S^4 \) has dimension (as a vector space) at most \( 1 \).

Proof: If the linear system is a net (or of smaller dimension), its base locus will be nonempty.
8. CUBIC THREEFOLDS ON $\mathcal{A}_4 \setminus (\mathcal{I}_4 \cup \mathcal{A}_{n11})$

We wish to prove what has been announced in 1.8. Using the results of section 7, we first define the cubic threefold for every $A \in \mathcal{A}_4 \setminus (\mathcal{I}_4 \cup \mathcal{A}_{n11})$.

(8.1) LEMMA: Let $A$ be in $\mathcal{A}_4 \setminus (\mathcal{I}_4 \cup \mathcal{A}_{n11})$. Then $P^{-1}(A)$ is two-dimensional.

Proof: The generic Prym-curve in $P^{-1}(A)$ is smooth. If the fiber of the Prym map is three-dimensional (or more), then there will be (at least) a two-dimensional family of singular Prym-curves in $P^{-1}(A)$:

The set of singular curves is a divisor and it intersects every $P^{-1}(A)$ in a curve by 7.1. At a generic intersection point the intersection $V$ of the Zariski tangent spaces to $P^{-1}(A)$ and the divisor of singular curves will have dimension 2 or 3. As we are supposing that $P^{-1}(A)$ has dimension $\geq 3$, (at least) a two-dimensional subspace of the infinitesimal deformations in $V$ will correspond to global deformations.

This is ruled out in 7.1. Q.E.D.

Recall that in the proof of 7.5 we defined $1_X$ for $X$ generic in a component of $P^{-1}(A)$ and showed that its base locus is two-dimensional. Also recall that $X_\lambda$ was defined in 2.4 for all smooth (nonhyperelliptic, but this is always true because of the hypothesis on $A$) curves $X$ and we saw in the proof of 7.5 that for $X$ generic in a component of $P^{-1}(A)$ the base locus of $1_X$ is the union of $\Sigma(X)$ and $\Sigma(X_\lambda)$.

(8.2) PROPOSITION: Let $A$ be in $\mathcal{A}_4 \setminus (\mathcal{I}_4 \cup \mathcal{A}_{n11})$. Then the pencil $1_X$ is defined for all $X \in P^{-1}(A)$.

Proof: The image of $\Sigma(X)$ by $\tilde{h}$ is contained in a plane for all $X$ because it is so generically. Also, $\tilde{h}(\Sigma(X))$ is not a line because it contains the image under $\tilde{h}$ of $\chi X$. 
By the analysis in 7.1 and 7.8, \( \tilde{h}(\chi X) \) cannot be a line or several times a line. Q.E.D.

**DEFINITION**: For \( A \) in \( G_4 \setminus \left( \tilde{g}_4 \cup G_{n11} \right) \) define \( T \) to be the smallest variety containing all the lines \( l_X \) for \( X \in P^1(A) \).

As the image of \( \Sigma(X) \) in \( \mathcal{H}^{0} \) spans a plane, no divisor in \( |\mathcal{H}^{0} \setminus l_X \) contains \( \Sigma(X) \). As in the proof of 7.5, for smooth \( X \), we still have the projection from \( l_X : \mathcal{H}^{0} \to N \). The lines corresponding to curves tetragonally related to \( X \) go to points on the plane quintic \( Q \). Notice that as we have a two-dimensional family of lines \( l_X \), a generic hyperplane section of \( T \) contains a finite number of these lines. From the generic case we deduce that the number of distinct lines \( l_X \) in a generic hyperplane is less than or equal to 27. We have:

**(8.3) LEMMA**: A generic hyperplane section of \( T \) contains exactly 27 lines \( l_X \).

**Proof**: Let \( H \) be a generic hyperplane in \( \mathcal{H}^{0} \) and suppose that \( H \) contains less than 27 lines \( l_X \). Then there is a line \( l = l_Y \) (for a smooth \( Y \) because \( H \) is generic) in \( H \) which counts twice.

We can find a one-parameter family \( \{A_t\} \) of ppav's with central fiber \( A_0 = A \) such that the generic member of \( \{A_t\} \) is a generic ppav. For such a family we can find a family of generic hyperplanes \( H_t \) with \( H_0 = H \) and a family of pairs of lines \( \{l_{1t}, l_{2t}\} \) in \( T_t \cap H_t \) such that for \( t \neq 0 \), \( l_{1t} \) and \( l_{2t} \) are distinct and \( l_{10} = l_{20} = 1 \). Let \( l_{1t} = U_t \), \( l_{2t} = l_{V_t} \) and let \( l_{3t} = l_{X_t} \) be a line incident to both \( l_{1t} \) and \( l_{2t} \) for all \( t \). If \( l_{1t} \) and \( l_{2t} \) meet for \( t \) generic suppose in addition that \( l_{3t} \) does not pass through the intersection point of \( l_{1t} \) and \( l_{2t} \).

i) First suppose that the lines \( l_{1t} \) and \( l_{2t} \) do not meet for \( t \neq 0 \). Then they span \( H_t \) for each \( t \neq 0 \). If the three lines \( l_{1t}, l_{2t}, l_{3t} \) come together when \( t \) goes to 0, replace \( l_{2t} \) by \( l_{3t} \) and reason as in ii) below.

Otherwise, when \( l_{1t} \) and \( l_{2t} \) come together, the two \( g_4^{1t} \)'s relating \( X_t \) to \( Y_t \) and \( U_t \) come together or become opposite because the intersection of \( \Sigma(Y) \) with \( \Sigma(X) \)
is the set of pairs projecting to pairs \((p,q) \in X^{(2)}\) such that \(h^0(g^1_d - p - q) > 0\) (reason on \(\tilde{X}^{(2)}\) and \(\tilde{Y}^{(2)}\) as in the proof of 7.5). Hence, as in the proof of 5.20, the Del-Pezzo surface \(q_{1t} \cap q_{2t}\) acquires a double point. Here \(q_{1t}\) and \(q_{2t}\) are the quadrics of rank 4 in the canonical space of \(X_t\) associated to the two \(g^1_d\)'s. However the generic quartic Del-Pezzo containing \(kX\) is smooth. (We are using the fact that a generic Prym-curve in \(P^1(A)\) is nonsingular.)

ii) Now suppose that for all \(t \neq 0\), \(l_{1t}\) and \(l_{2t}\) meet. Then by 5.12 we can suppose that \(X_t, Y_t\) and \(U_t\) are tetragonally related. At the limit, \(X = X_0, Y = Y_0\) and \(U = U_0\) are smooth as \(H\) is generic and they are also tetragonally related. However \(l_Y = l_U = 1\) so we have three possibilities:

a) \(Y = U\). For any tetragonal triple there is a trigonal curve \(V\) of genus 6 trigonally related to all three curves and the three points of order 2 on \(JV\) for the three trigonal constructions are the three nonzero elements of a totally isotropic rank 2 vector space in \((JV)_2\). For each \(g^1_d\) on \(Y\) there is exactly one pair \(\{\text{trigonal curve, point of order 2}\}\) trigonally related to \(Y\) through that \(g^1_d\). So the points of order 2 in \(JV\) corresponding to \(Y = U\) must coincide. Then the sum of these points of order 2 is equal to the point of order 2 corresponding to \(X\). So the point of order 2 corresponding to \(X\) is 0: this is not possible because then \(V\) would be isomorphic to \(X\) and the \(g^1_d\) on \(X\) will have a base point.

b) \(Y = Y_λ\) and \(U = U_λ\). By a) we can suppose that \(Y \neq X\). So, as \(Y\) determines its \(g^1_d\) as in i), this \(g^1_d\) is a theta-null on \(X\) and this is not the case for a generic \(g^1_d\).

c) \(Y = U_λ\). As in a) \(Y \neq X\) and \(U \neq X\). The curve \(X\) is tetragonally related to \(U_λ\) through \(K_X - g^1_d: \) we conclude as in b).

Q.E.D.

(8.4) Let \(\tilde{\mathcal{A}}\) be the blow up of the universal family \(\mathcal{A} \to \mathfrak{S}\) along the zero section. Let \(\tilde{\mathcal{A}}\)' be its restriction to the locus \(\mathfrak{C}_d \backslash (\mathfrak{C}_d \cup \mathfrak{C}_{n11})\). By section 7, the \(\Gamma_{00}\) maps define
a morphism
\[ \tilde{\mathbb{A}}' \to \mathbb{P}^*_{\mathcal{A}_4} \setminus \tilde{\mathcal{I}}_4 \cup \mathcal{O}_{11} \]
which is generically finite on each \( \mathbb{A} \) and the branch locus of this morphism intersects each fiber \((2\Theta'_0)\) in a divisor. One component of the branch locus is the image of the exceptional locus \( \mathcal{E} \) in \( \tilde{\mathbb{A}}' \). From the generic case we deduce that the second component of the branch locus is the set of hyperplanes which contain less than 27 lines.

If \( T \) is a union of planes, then by 8.3, it will have to be a union of 27 distinct planes. Then the set of hyperplanes containing less than 27 lines will be exactly the set of hyperplanes containing one of the planes of \( T \): this is impossible by the above discussion.

So \( T \) is a hypersurface and by continuity the degree of \( T \) is less than or equal to 3. As the plane quintics \( Q \) are reduced we have that \( T \) is neither a hyperplane nor a union of hyperplanes. So \( T \) can at worst degenerate to a quadric or the union of a quadric and a hyperplane.

(8.5) LEMMA: \( T \) is an irreducible cubic threefold.

Proof: By 8.3, a generic hyperplane section of \( T \) contains exactly 27 distinct lines \( l_X \). These must intersect at least as much as the 27 lines in a generic hyperplane section of the cubic threefold associated to a generic abelian variety.

Consider a generic line \( l_X \) and consider the family of pairs of lines corresponding to pairs of curves tetragonally related to \( X \) through the same \( g^1_4 \). Generically, by 8.3, each of these is a pair of distinct lines, distinct from \( l_X \) and together with \( l_X \) it gives a triple of distinct lines in a plane section of \( T \). Again by 8.3, a generic such plane section of \( T \) does not contain a one-dimensional family of lines \( l_X \) and contains exactly three distinct lines. We first deduce from this that \( T \) is not a quadric because otherwise each of these planes would be contained in \( T \) and as there is at most a one-dimensional family of planes in a nondegenerate quadric each plane would have to contain a one-dimensional
family of lines $l_X$.

Suppose now that $T$ is the union of a quadric and a hyperplane $H_0$. Choose a
generic hyperplane $H$. Out of each triple of lines in a plane in $H$, one has to be in $H_0$.
Then $H \cap H_0$ would be a plane containing more than 3 lines $l_X$, so it contains a
one-dimensional family of them. However this cannot be true for a generic hyperplane $H$
because otherwise $P^{-1}(A)$ would be more than two-dimensional. Q.E.D.

(8.6) **PROPOSITION**: $T$ is a cubic threefold with at most isolated double points as
singularities.

*Proof*: Suppose that the cubic threefold $T$ has a triple point $t$. Then $T$ is the cone of
vertex $t$ over an irreducible cubic surface $S$. The family of lines in $T$ has the following
components: lines through $t$ and lines in the planes projecting to lines in $S$. A generic
hyperplane section of $T$ is isomorphic to $S$. So by 8.3, $S$ is smooth. Then the lines $l_X$
will be in the planes above the lines in $S$: $T$ would be a union of planes.

Suppose now that $T$ has a double curve $C$. Then every hyperplane section of $T$
will be singular. Also, as $T$ is irreducible, a general hyperplane section of $T$ will contain
a finite number of lines and as it is singular it will contain less than 27 lines. This is ruled
out by 8.3. Q.E.D.

From 8.6 we deduce in particular that, as in the generic case, $T$ has a structure of
conic bundle over $\mathbb{P}^2$ with discriminant curve $Q$ for each Prym-curve $X$. We have

(8.7) **THEOREM**: $T$ is singular if and only if $A \in A_4 \setminus (A_4 \cup A_{n11})$ has a vanishing
theta-null.

For the proof we first need

(8.8) **LEMMA**: If $X$ is smooth, then a vanishing theta-null $E$ on $X$ such that $h^0(\eta(E))$
is odd corresponds to a double point on $X_\lambda$.

*Proof*: Parametrize $A$ with $X$. Let $\pi: \tilde{X} \to X$ be the projection. Recall that $\tilde{X}_\lambda$ is the
set of embeddings of $\tilde{X}$ in $\Theta$, i.e., $\tilde{X}_\lambda$ can be identified with
\{ E' \in \text{Pic}^8 \bar{X} : \pi_+ E' = K_\bar{X} , \ h^0(E') \text{ is odd} \}.

Let \( E' = \pi^* E \), then \( E' \) verifies
\[
\pi_+ E' = 2E = K_X , \quad h^0(E') = h^0(E) + h^0(\pi_+(E)) , \quad E' = K_\bar{X} - E' = \sigma E'
\]
As \( h^0(E) \) is even, we see that \( h^0(E') \) is odd. So \( E' \) is an element of \( \bar{X}_\lambda \). The covering involution \( \sigma_\lambda \) on \( \bar{X}_\lambda \) is induced by the involution \( E' \mapsto K_\bar{X} - E' \) on \( \text{Pic}^8 \bar{X} \).

So the cover \( \pi_\lambda : \bar{X}_\lambda \to X_\lambda \) is ramified at \( E' \). So \( X_\lambda \) is singular.

**Proof of theorem 8.7** : \( T \) is singular if and only if, for all \( X \), \( Q \) is singular. Equivalently, for all \( X \), \( X \) has a vanishing theta-null because the generic curve in \( P^1(A) \) is smooth by 7.1. Hence the locus of singular cubic threefolds is contained in the image by \( P \) of the one vanishing theta-null locus \( \theta_{\text{null},5} \) in \( \mathcal{P}_5 \setminus P^{-1}(\mathcal{Q}_4 \cup \mathcal{Q}_{n11}) \). This has dimension 11.

By [B1], \( \theta_{\text{null},5} \) has exactly two irreducible components \( \theta_{\text{null}1} \) and \( \theta_{\text{null}2} \). These are described as follows. Let \( X \in \theta_{\text{null},5} \) be generic. Let \( g_4^1 \) verify \( 2g_4^1 = K_X \).

Then
- \( X \in \theta_{\text{null}1} \) if and only if \( h^0(\pi^* g_4^1) \) is even,
- \( X \in \theta_{\text{null}2} \) if and only if \( h^0(\pi^* g_4^1) \) is odd.

By [B1], \( P^{-1}(\theta_{\text{null}}) \) is the closure of \( \theta_{\text{null}1} \) and \( \theta_{\text{null}} \) is the closure of \( P(\theta_{\text{null}1}) \). By 8.8 and 7.1, \( \theta_{\text{null}2} \) maps onto \( \mathcal{Q}_4 \setminus (\mathcal{Q}_4 \cup \mathcal{Q}_{n11}) \) with one-dimensional fibers everywhere.

It follows that \( T \) is singular on \( \theta_{\text{null}} \) only.

Q.E.D.
9. (SINGULAR) CUBIC THREEFOLDS
OVER $g_4$

We wish to define the cubic threefold for the jacobian of a smooth nonhyperelliptic curve of genus 4.

Let $A = JC$ be the jacobian of a nonhyperelliptic curve. By [B1], $P^{-1}(A)$ has two components interchanged by $\lambda$, namely, those described in 4.2. In particular, $P^{-1}(A)$ is two-dimensional.

By [W3], the base locus $V(\Gamma_{00})$ of $|2\Theta|_{00}$ is equal to $C - C \cup \{\pm(K_C - 2g_4^1)\}$.

Using 4.3 it is easily seen that

$$\Sigma(C_{pq}) = C - C \cup W_2 - p - q \cup p + q - W_2.$$  

Also, for a trigonal $X \in P^{-1}(A)$, associated to $g_4^1$ on $C$:

$$\Sigma(X) = \{s + t - s' - t' : h^0(g_4^1 - s - t) > 0 \text{ and } h^0(g_4^1 - s' - t') > 0\}.$$  

From this we deduce that we can define the pencil $l_X = l_{X,\lambda}$ as in 7.5. As in the proof of 7.5, the base locus of each pencil will be two-dimensional because otherwise the base locus of $|2\Theta|_{00}$ will be a divisor. So we can define $T$ to be the reduced subvariety of $|2\Theta|_{00}$ containing the lines corresponding to the Prym-curves for $A$ ($T$ is irreducible because $P^{-1}(A)$ has two irreducible components exchanged by $\lambda$).

We still have the projection from $l_X : |2\Theta|_{00} \to N$. (because the image of $\Sigma(X)$ (resp. $\Sigma(C_{pq}) \setminus C - C$) by $h$ is a plane for a trigonal $X$ (resp. singular $C_{pq}$)). Under this projection, the lines corresponding to Prym-curves tetragonally related to $X$ go to points on the plane quintic $Q$ parametrizing singular quadrics through the canonical model of $X$.

So $T$ cannot be a plane neither a hyperplane. $T$ is a threefold and by continuity the degree
of $T$ is less than or equal to 3.

Recall that in 4.5 we computed the number of lines in a generic intersection of translates of $\Theta$ for a jacobian. The computation works for the jacobian of a smooth nonhyperelliptic curve. Hence a generic hyperplane contains exactly 27 distinct lines. These intersect at least as much as in the case where $A$ is generic.

Then, using the incidence configuration of the lines in a generic hyperplane, one sees that if $T$ is a quadric, then the lines cannot be all distinct. Also the lines cannot be distinct unless the hyperplane section of $T$ is a smooth cubic surface.

Also, all the lines cannot pass through the same point by the surjectivity of $\Sigma_A \to A$. So $T$ has no triple point.

We have proved:

(9.1) **PROPOSITION**: $T$ is a cubic threefold with at worst a finite number of double points.

From the above it also follows that $T$ has conic-bundle structures over the nets $N$ with discriminant curves $Q$. As $X$ (generic in $P^1(A)$) is smooth and trigonal, $Q$ is isomorphic to $X$ with two points identified and has exactly one double point. It follows that

(9.2) **PROPOSITION**: $T$ has exactly one double point. In particular, every line in $T$ is of the form $l_X$ for some Prym-curve $X$.

We still have the double cover $1: \tilde{T} = P^1(A) \to F$.

Recall that the two components of $P^1(A)$ are each isomorphic to $C^{(2)}$. Let $\xi$ be one of the two (possibly equal) $g^1_3$'s on $C$ and let $\xi' = K_C - \xi$. The two components intersect in the set of singular curves $C_{pq}$ such that $h^0(\xi - p - q) > 0$ or $h^0(\xi' - p - q) > 0$. Suppose, for instance, that $h^0(\xi - p - q) > 0$. Then $\xi$ induces a $g^1_3$ on $C_{pq}$. Hence, by continuity (from the smooth trigonal case), the plane quintic $Q_{pq}$ associated to $C_{pq}$ is isomorphic to $C_{pq}$ with two points identified. However $Q_{pq}$ is also isomorphic to
the curve \( C' \) parametrizing the set of lines through the double point of \( T \) with two pairs of points identified. Hence \( C' = C \).

(9.3) **Lemma:** \( C \) parametrizes the set of lines through the double point of \( T \).

(9.4) **Corollary:** The quadric tangent cone to \( T \) at its double point has rank \( \geq 3 \).

**Proof:** This is because, by continuity from the case where \( T \) has an ordinary double point, the embedding of \( C \) in the exceptional \( \mathbb{P}^3 \) above the double point of \( T \) is the canonical embedding of \( C \).

(9.5) **Remark:** If we had known that \( T \) has an ordinary double point, we could have deduced the above from the fact that \( F = C^{(2)} \) and from the results of [CG].
10. DOUBLE SOLIDS OVER $\mathfrak{g}_4$ AND $\theta_{\text{null}}$

Generically over $\mathfrak{g}_4$ and $\theta_{\text{null}}$, because the generic plane quintics $Q$ have one ordinary double point, $T$ has exactly one ordinary double point.

Consider the map $E : Z_A \to K(A) = A_{\pm 1}$ which to each double solid $Z$ associates the element $\bar{a}$ of $K(A)$ ($a \in A$) with the property $\Theta \cdot \bar{a} = E_Z$ the Fano variety of lines in $\tilde{Z}$. For a generic ppav, by 3.5, the map $E$ is of degree 16 onto the image of $\mathcal{E}$ in $K(A)$. The image of $\mathcal{E}$ by $\tilde{h}$ is the set of hyperplanes of $|2\Theta|_0$ which contain less than 27 lines. Equivalently, the image of $Z_A$ by the composition $D^* = \tilde{h} \cdot E$ is $T^*$. As $T$ is smooth, it is birationally equivalent to $T^*$ ([CG]). The composition of $D^*$ with the birational equivalence $T^* \to T$ is just $D : Z \mapsfrom D_Z \in |2\Theta|_0$.

Generically on $\mathfrak{g}_4 \cup \theta_{\text{null}}$, the set of hyperplanes in $|2\Theta|_0$ which contain less than 27 lines is equal to $T^* \cup H_0$ where $H_0 \equiv \mathbb{P}^3$ is the set of hyperplanes in $|2\Theta|_0$ through the double point $t$ of $T$. If $A$ is in $\theta_{\text{null}}$, put the double point of $\Theta$ at the origin. We have

(10.1) **PROPOSITION**: The pullback of $H_0$ is the divisor of zeros of $\Theta^2$ in the one vanishing theta-null case and $\theta_\xi \cdot \theta_\xi$ in the jacobian case.

(Recall that $\theta$ is a nonzero section of $\Theta_A(\Theta)$ and $\theta_\xi$ is a nonzero section of $\Theta_A(W_3 - \xi)$ in the curve case, $\xi$ being a $g^1_3$ on the curve, $W_3$ the variety of effective divisor classes in $\text{Pic}^3 C$.)

**Proof**: On $\theta_{\text{null}}$ we know that $T^* \cup H_0$ is contained in the branch locus of $\tilde{h}$ because it is the specialization of the generic $T^*$.

Generically on $\mathfrak{g}_4$, consider a generic singular Prym-curve $C_{pq}$. Consider a
generic pencil of hyperplanes in $|2\Omega|_{00}$ containing the line associated to $C_{pq}$ and with special member a hyperplane tangent to $T$ at a unique point (or a generic hyperplane through the double point of $T$) which is not on the line of $C_{pq}$. Each of these hyperplanes projects to a line in the net $N$ of quadrics containing the canonical model of $C_{pq}$. We see that the special hyperplane projects to a line $1$ which is simply tangent to the plane quintic $Q$ in $N$ (or is a generic line through the double point of $Q$). Hence the Del-Pezzo surface which is the intersection of the quadrics in $1$ has one node and as in the proof of 5.15 we see that the number of preimages by $\tilde{h}$ of the special member of the pencil is less than the number of preimages of a general member. So we see that also on $J_4^\ast, T^\ast \cup H_0$ is contained in the branch locus of $\tilde{h}$.

So the inverse image of $H_0$ in $A$ is split. Now the proposition follows from the fact that the Picard group of $JC$ has rank 1 for $JC$ generic in $\theta_{null} \cap J_4$. Q.E.D.

We are first going to determine the double solids (or degenerations of double solids) with intermediate jacobian a generic element of $\theta_{null}$.

By [C1], when the equation of the branch locus $B$ of $Z$ is a second degree homogeneous form in the second degree forms vanishing on the double points $p_i$ of $Z$, then $JZ$ has a vanishing theta-null. The moduli space of these double solids has dimension 12, hence the generic element of $\theta_{null}$ is obtained in this way. Also $B \to \rho(B)$ is of degree 2 and $\rho(B) = 2q_0$ where $q_0$ is a quadric ([C1]). Also, by [C1], the Prym-curves associated to $Z$ have two vanishing theta-nulls. So their plane quintics have two double points. As $T$ is generic with one ordinary double point it follows that the set $(Z_A)_0 \subset Z_A$ of these double solids is blown down to the singular point of $T$ by the map $D$. Hence

(10.2) PROPOSITION : $D^\ast((Z_A)_0) = H_0$.

Conversely, it is immediately seen that, for a double solid $Z$, $\rho(B) = 2q_0$ only if $B$ is a quadric in the quadrics through the $p_i$'s.
(10.3) **COROLLARY:** \( q_0 \) is the tangent cone to \( \Theta \) at its (unique because \( A \) is a generic element of \( \theta_{\text{null}} \)) double point.

**Proof:** Use 10.1 and 10.2.

Let \( X \in P^1(A) \) be generic: \( X \) is a generic element of \( \theta_{\text{null}} \). Let \( g = g^1_4 \) be the unique theta-null of \( X \). Let \( \{p,q\} \in X^{(2)} \) be a pair such that the image \( X^{pq} \) of \( X \) in \( \mathbb{P}^2 \) given by \( g^2_6 = K_X - p - q \) has an everywhere tangent conic and is otherwise generic.

Then (2.2) there is a unique double solid \( Z \) such that \( \tilde{X} \) parametrizes the lines in \( Z \) through \( p_1 \). By adjunction, the sums \( D_1 + D_2 \), where \( D_1 \in g \) and \( D_1 \in K_X - g \), are cut on \( X^{pq} \) by a conic (= \( \mathbb{P}^1 \)) of cubics through \( p_2, ..., p_6 \). As \( g = K_X - g \), all these cubics are everywhere tangent to \( X^{pq} \). Conversely any such family of cubics cuts, residually on \( X^{pq} \), a \( g^1_4 \) such that \( 2g^1_4 = K_X \).

For cubics, being singular is one condition. Hence in a conic of cubics as above there are at least a finite number of singular cubics. Let \( C'' \) be a singular such cubic. The geometric genus (i.e., the genus of the normalization) of \( C'' \) is 0. Let \( \tilde{C}'' \) be the inverse image of \( C'' \) in \( B \). This has degree 12 because it is the complete intersection of \( B \) and the cone over \( C'' \) with vertex \( p_1 \). As \( C'' \) is everywhere tangent to the branch locus of \( B \to \mathbb{P}^2 \), the map \( \tilde{C}'' \to C'' \) is unramified and \( \tilde{C}'' \) breaks into two components \( C''_1 \) and \( C''_2 \). Each of these is a space sextic. By [C2] these curves are determinantal and the quartics containing them are determinantal.

Conversely suppose that \( B \) is determinantal. Delete one row of the determinant defining \( B \), then the three by three minors that are left define a \( \mathbb{P}^2 \) of cubics in \( \mathbb{P}^3 \) whose base locus is a sextic \( C' \) of genus 3 on \( B \) containing the double points of \( B \). Each of the cubics containing \( C' \) cuts, residually on \( B \), another sextic of genus 3. We see that \( B \) contains a \( \mathbb{P}^3 \) of sextics of genus 3 (see [C2]) which all pass through the double points of \( B \).

By [C2], \( \Theta \) is parametrized by the set of sextics in \( Z \) with a triple point at \( p_1 \).
and passing through $p_i$ for $i > 1$. Each such sextic $C'$ projects in $\mathbb{P}^2$ to a cubic $C''$ through $p_2, \ldots, p_6$. The lift, determined by $C'$ via incidence, of the canonical divisor cut by $C''$ on $X^p_q$ gives the image of $C'$ by the Abel-Jacobi map $AJ$.

Having a triple point at $p_1$ is two conditions for the $\mathbb{P}^3$ of sextics contained in $B$. Hence there is a $\mathbb{P}^1$ of such sextics $C'$. The image of such a $C'$ in $\mathbb{P}^2$ is a cubic $C''$ everywhere tangent to $X^p_q$ because the inverse image of $C''$ in $B$ has two components.

Notice that, a posteriori, we obtain the fact that if $X^p_q$ has a conic of everywhere tangent cubics through its double points, then the inverse images of these cubics in $B$ are split.

We have proved:

(10.4) **PROPOSITION**: The branch loci of the double solids above $T^* \setminus H_0$ are determinantal.

We turn to jacobians. Let $Z$ be a double solid with a unode $p_0$ (see 1.8) and three ordinary double points $p_1, p_2, p_3$ ($Z$ generic for these properties). Donagi observes:

(10.5) **PROPOSITION**: $JZ$ is the jacobian of a smooth curve of genus 4.

**Proof**: Let $x, y, z, w$ be coordinates on $\mathbb{P}^3$ such that $p_0 = (0,0,0,1)$ and near $p_0$, $B$ has the expansion

$$B = x^2 + (y - az)(y - bz)(y - cz) + \text{higher order}.$$ 

We can also suppose that $p_1 = (1,0,0,0)$, $p_2 = (0,1,0,0)$, $p_3 = (0,0,1,0)$. The plane representation of $X_1$ has a triple point (with two other double points) and $X_1$ is trigonal. This is because if $X_1$ is given by the equation $f$ locally near $p_0$, then $B$ has local equation $x^2 - f = 0$ near $p_0$. The Prym of a trigonal curve is always a jacobian $[R]$.

Q.E.D.

(10.6) Donagi also observes that the discriminant curve $X_0$ for the projection from $p_0$
has 6 double points. Three of them are colinear: these are the infinitely near double points \( p_4, p_5, p_6 \) in the tangent cone to \( B \) at \( p_0 \). Normalizing \( X_0 \) at five of its double points and keeping \( p_i \) (4 \( \leq \) i \( \leq \) 6) one obtains the discriminant curve \( X_1 \). These all have the same normalization \( C \) and \( A = JZ = JC \):

In \( \overline{\mathbb{P}}^3 \) the exceptional quadric \( Q_0 \) above \( p_0 \) is the union of two projective planes meeting in a line: the tangent cone to \( B \) at \( p_0 \). Hence, as expected, the double cover of \( X_0 \) in \( Q_0 \) is split. Its two components are smooth (hence isomorphic to \( C \)) and meet in the following way: let \( p_i \) be obtained by identifying \( p'_i \) and \( p''_i \) on \( C \), then \( p'_i \) on one component is identified with \( p''_i \) on the other. So the curves \( \tilde{X}_i \) are of the singular type in 4.2 for \( i \geq 4 \).

We claim that the only double solids with intermediate jacobian the jacobian of a curve \( C \) are the unodal ones or the images of the unodal ones by one of the Cremona transformations described in 1.9. Let \( X \) be a smooth trigonal curve in \( P^1(JC) \).

(10.7) LEMMA: Every plane representation of \( X \) of degree 6 has a triple point.

Proof: Choose a \( g^2_6 \) on \( X \). We need to show that \( h^0(g^2_6 - g^1_3) > 0 \). Let \( p \) and \( q \) be two points on \( X \) such that \( h^0(g^2_6 - p - q) = 2 \). Put \( g^1_4 = g^2_6 - p - q \). The variety \( W^1_4 \) of \( g^1_4 \)'s on \( X \) is equal to

\[
g^1_3 + X \cup K_X - g^1_3 - X
\]

(see [W3] or [ACGH] or [AM]). If \( g^1_4 = g^1_3 + t_0 \) for some \( t_0 \) in \( X \), we are done. Otherwise there exists \( t_0 \in X \) such that \( g^1_4 = K_X - g^1_3 - t_0 \).

We have \( K_X - g^2_6 = g^1_3 + t_0 - p - q \) is effective by Riemann-Roch. As \( X \) is nonhyperelliptic, \( X \) has no \( g^2_4 \)'s by Clifford. Two cases are possible:

- \( h^0(g^1_3 - p - q) > 0 \). Let \( p+q+s \in g^1_3 \). Then \( g^1_4 - s = g^2_6 - p - q - s = g^2_6 - g^1_3 \) is effective.

- \( t_0 = p \) or \( q \). Say \( t_0 = q \). Then \( g^2_6 = K_X - g^1_3 + p = g^2_5 + p \). And \( g^2_5 - g^1_3 = K_X - 2g^1_3 + p \) is effective because \( K_X - 2g^1_3 \) is so.
(10.8) COROLLARY: For any plane representation of $X$ with an everywhere tangent conic, the double cover of $\mathbb{P}^2$ branched along $X$ has a unode or is the image of a double solid with a unode by a Cremona transformation.

Proof: Letting the Cremona group act, we can suppose that a discriminant curve $X_1$ is trigonal. Hence by 10.7, $X_1$ has a triple point in $\mathbb{P}^2$. Then, as in 10.5, it is seen on the local equations for $B$ and $X_1$ that $B$ has a unode.

(10.9) PROPOSITION: If $Z$ has two unodes, then $JC \in \Theta_{\text{null}} \cap \mathfrak{Q}_4$, i.e., $2\xi = K_C$.

Proof: Suppose that $Z$ has two unodes and is generic for this property. Then the discriminant curve $X_0$ has one triple point and three ordinary double points which are colinear. Let $g_0^2$ be associated to this plane representation of $C$ and write $g_0^2 = g_3 + t_1 + t_2 + t_3$ (this corresponds to the triple point). Suppose, for instance, that $g_3 = \xi$.

Write $K_C - t_1 - t_2 - t_3 = t_4 + t_5 + t_6$. And $\xi - t_i = s_i + u_i$ for $i = 4, 5, 6$.

Then

$$g_0^2 - s_i - u_i = t_1 + t_2 + t_3 + t_i = K_C - t_j - t_k$$

with \( \{t_1, t_2, t_3\} \). Thus these give the three double points of $X_0$. The colinearity condition means that $s_4 + u_4 + s_5 + u_5 + s_6 + u_6 \in g_0^2$. Hence

$$s_5 + u_5 + s_6 + u_6 = g_0^2 - s_4 - u_4 = K_C - t_5 - t_6$$

and

$$K_C = t_5 + s_5 + u_5 + t_6 + s_6 + u_6 = 2\xi.$$

(10.10) We wish to determine the double solids $Z$ which verify $D_Z = t$ the singular point of $T$.

We have $D_Z = t$ if and only if the lines $l_i$ corresponding to the discriminant curves $X_i$ of $Z$ all pass through $t$. That is they correspond to elements \( \{s, t\} \in C^{(2)} (= \text{the minimal desingularization of } F) \) such that $h^0(\xi - s - t) > 0$ or $h^0(K_C - \xi - s - t) > 0$.

For instance let $X_1 = C_{st}$ with $h^0(\xi - s - t) > 0$. Then $\xi$ induces a $g_3^1$ on $C_{st}$. As $C_{st}$ is in the closure of the locus of smooth trigonal curves, the plane representation $X_1$ of
$C_{st}$ has a triple point by 10.6. Hence, as before, the double cover of $\mathbb{P}^2$ branched along $X_1$ has a unode. The geometric genus of $X_0$ being 4, in the most generic case, $X_0$ has three ordinary double points besides the triple point.

Let $q_0$ be the exceptional quadric in $\tilde{Z}$ above the double point $p_0$ corresponding to $C_{st}$. $q_0$ is the double cover of $\mathbb{P}^2$ branched along $C_{0t}$. We investigate the condition that the double cover of $X_0$ in $q_0$ is split (see 4.2). Let

$$C_{0t}X_0 = 3.(\sum_{1 \leq i \leq 6} q_i)$$

then, by [C1] (and by continuity because the statement is true generically), the double cover $\tilde{C}_{st} \rightarrow C_{st}$ obtained from $Z$ is given by the point of order 2

$$\left| g_6^2 - \sum_{1 \leq i \leq 6} q_i \right| .$$

The pull-back of this to $C$ is trivial: $g_6^2 = \left| \sum_{1 \leq i \leq 6} q_i \right|$ (on $C$) and $C_{0t}$ is twice a line. As $C_{0t}$ is the tangent cone to $B$ at $p_0$, the point $p_0$ is a unode for $B$ (the unode is the simplest singularity for which the tangent cone is twice a line). By 10.9, $C$ has a vanishing theta-null.

The other cases in which all the Prym-curves are singular are specializations of this one. Thus

There are no "double solids" above $t$. 
11. TORELLI

Let $A$ be a generic ppav. Let $Z \in \mathcal{Z}_A$ be a double solid with intermediate jacobian $A$. With the notations of 1.9, recall that to $Z$ is associated a Kummer variety $K$ with a distinguished double point $t_Z$. Also recall that if $x \in A$ is such that $\Theta_x = E_Z$ is the Fano variety of lines of $Z$, then the $\Gamma_0$-map $\tilde{h}$ is ramified at $x$.

(12.1) **PROPOSITION**: The (projectivised) kernel of the differential of $\tilde{h}$ at $x$ is $t_Z$, after we identify $T_xA$ with $T_0A$ by translation.

**Proof**: The kernel of the differential of $\tilde{h}$ at $x$ is the intersection of the tangent spaces at $x$ to the $\Gamma_0$-divisors which contain $x$. These divisors are the elements of $H = \tilde{h}(x)$ which is the tangent hyperplane to $T$ at $D_Z$. Recall that there are 32 double solids with associated $\Gamma_0$-divisor $D_Z$, we pick one of these: $Z$. View $D_Z$ as an element of $(\Omega_{\Gamma_0})^*$, then $D_Z$ is the tangent hyperplane to $T^*$ at $H$. Hence, as $T^*$ is in the branch locus of $\tilde{h}$, $D_Z \subset A$ is singular at $x$.

Now, view $D_Z$ as an element of $\Omega_{\Gamma_0}$, more precisely, of $T$. The hyperplane $H$ is generated by the six lines $l_i$ in $T$ passing through $D_Z$. For a generic choice of the 32 double solids above $D_Z$, as $x$ is a simple ramification point, the kernel of the differential of $h$ at $x$ is one-dimensional. Hence the only $\Gamma_0$-divisor singular at $x$ is $D_Z$, and so, given a pencil $l_i$, the tangent space to the $D \in l_i \setminus \{D_Z\}$ is the same three-dimensional vector space. We are going to find the intersection of these.

Let $X$ and $X_\lambda$ be associated to $l_1$. Then $\tilde{X}$ or $\tilde{X}_\lambda$ parametrizes lines in $Z$ through $p_1$, suppose, for instance, that $\tilde{X}$ does. Then $\tilde{X}_\lambda$ parametrizes lines in $tZ$ through $p_1$. Suppose that we project $Z$ from $p_1$ and the plane representation of $X$ is
given by \( \mathfrak{K}_X - s - t \), then, for a choice of liftings of \( s \) and \( t \) in \( \tilde{X} \), say \( s', t' \), we have \( \mathbf{x} = \pm[s', t'] = s' + t' - s' - s' \) (see 2.3). Similarly \( \mathbf{x} = \pm[u', v'] \) where \( \mathfrak{K}_X - u - v \) gives the plane representation of \( X_\lambda \) obtained from \( Z_\lambda \). These plane representations live naturally in the exceptional plane \( P_1 \) above \( p_1 \) in \( \tilde{\mathbb{P}}^3 \). They have the same double points there because their double points are the images of \( p_2, \ldots, p_6 \) (still denoted by \( p_2, \ldots, p_6 \)). The images of \( s, t, u, v \) in \( P_1 \) lie on the conic through the points \( p_2, \ldots, p_6 \) ([C1]). So, in particular, \( s, t, u, v \) are not on a line in \( P_1 \). Recall that the map \( \rho \) is given by the linear system of quadrics through the \( p_i \)’s. The images of \( s, t, u, v \) by the extension of \( \rho \) are their Prym-canonical images.

Recall that \( \mathbf{x} \in \Sigma(X) \cap \Sigma(X_\lambda) \). For every \( D \in I_1 \), \( D \) contains \( \Sigma(X) \cup \Sigma(X_\lambda) \). Hence \( \mathbb{P}T_x \) contains \( \mathbb{P}T_x \Sigma(X) + \mathbb{P}T_x \Sigma(X_\lambda) = \langle s, t \rangle + \langle u, v \rangle \subset \mathbb{P}T_0 A \). By the above \( \langle s, t \rangle + \langle u, v \rangle \) is a plane and is equal to the image (still denoted by \( P_1 \)) of \( P_1 \) in \( \mathbb{P}T_0 A \) by the extension of \( \rho \).

Hence the common tangent plane at \( \mathbf{x} \) to the divisors in \( I_1 \) is \( P_1 \). And similarly, for all \( i \), the common tangent plane at \( \mathbf{x} \) to the divisors in \( I_i \) is \( P_i : \)

\[
\begin{array}{c}
P_i \leftrightarrow \tilde{\mathbb{P}}^3 \supset \tilde{B} \\
\searrow \downarrow \\
\mathbb{P}^3 = \mathbb{P}T_0 A
\end{array}
\]

By 1.9 the intersection of the \( P_i \)'s is \( t_Z \). Q.E.D.
12. MODULI OF SIX POINTS

Let $A$, $Z$, etc be as above. In $l2\Theta_{00}$ project from $D_Z$, this exhibits $T$ as a double cover of $\mathbb{P}^3$ branched along a quartic surface with six double points $q_i$ on a conic $C_{T,D}$: the images of the lines $l_i$ on the image of the tangent cone at $D_Z$ to $T \cap H$.

(12.1) PROPOSITION: The moduli of the points $q_i$ on $C_{T,D}$ is equal to the moduli of the points $p_i$ in $\mathbb{P}^3$.

Proof: Letting $\mathbb{P}GL(4,\mathbb{C})$ act, we can suppose that $p_1 = q_1$. Project from $p_1$, then the $q_i$'s ($i > 1$) project to the five intersection points of $Q_X$ with the image 1 of $C_{T,D}$. We are reduced to showing that if $q_2, \ldots, q_6$ are five singular quadrics in a pencil 1 of quadrics containing $\kappa X$, then the moduli of the $q_i$ in 1 is equal to the moduli in the conic $C_{s,t}$ containing them (or in $\mathbb{P}^2$) of the double points $p_2, \ldots, p_6$ of the plane representation $X^{st}$ of $X$ given by $\mathcal{K}_X - s - tl$.

Recall that $JX = P(Sing\Theta',Q)$. The map which to each $q \in Q$ associates its singular point in $\mathcal{K}_X^{l^*}$ is the Prym-canonical embedding $\chi_Q$ of $Q$ [B2]. Let $s_i$ be the singular point of $q_i$.

The curve $X^{st}$ is also the image of $\kappa X$ by the projection from the line $\langle s, t \rangle \subset \mathcal{K}_X^{l^*}$. Also $p_i$ is the image of $s_i$ by the projection from $\langle s, t \rangle$.

There exists a unique rational normal curve $R_{s,t}$ of degree 4 passing through $s_2, \ldots, s_6, s, t$. Moreover, as the image of $R_{s,t}$ in $\mathcal{K}_X - s - tl^*$ is the conic $C_{s,t}$ which also contains the images of $s$ and $t$, $R_{s,t}$ is tangent to $\kappa X$ at $s$ and $t$. Further, the moduli of the $p_i$'s on $C_{s,t}$ is equal to the moduli of the $s_i$'s on $R_{s,t}$.
Now consider the map $\chi : |K_X|^* = \mathbb{P}^4 \to (\mathbb{P}^2)^* = N^*$ given by the net $N$ of quadrics containing $\kappa X$. By [B2] the image of $\chi Q$ by $\chi$ is equal to the image of $Q \subset N$ by the dual map of $Q$. From the above considerations, it is immediately seen that $\chi(R_{s,t})$ is equal to the image of 1 by the dual map of $Q$ and that the $s_i$'s go to the images of the $q_i$'s. Q.E.D.
13. THETA-IDENTITIES

(13.1) Our theta identities are a translation of the scheme theoretic equality

\[ E_Z \cap (\Theta_c \cup \Theta_\wedge) = E_Z \cap D_Z \]

(for some \( c \in A \) and an appropriate choice of base point for the embedding of the fano variety of lines of \( \tilde{Z} \) in \( A \)) which we will prove below.

First notice that the 27 (section 4) Prym curves in \( E_Z \) are (see 4.8) \( \tilde{X}_i \), \( (\tilde{X}_i)_\lambda \)
and \( \tilde{X}_{ij} \) for all \( i \neq j \) between 1 and 6. We also know the two (see 3.2) translates of each curve inside \( E_Z \):

For the first 12 curves the two copies are obtained by adjoining to the strict transform of a line in \( Z \) (resp. \( \tilde{Z} \)) through \( p_i \) one of the two rulings of the exceptional quadric above \( p_i \) in \( \tilde{Z} \) (resp. \( \tilde{Z} \), see 3.5) (to obtain actual lines in \( \tilde{Z} \) (or \( \tilde{Z} \))). For the \( \tilde{X}_{ij} \), the two copies are obtained by subtracting one of the two lines in \( Z \) through \( p_i \) and \( p_j \) from the sum of each two lines giving us an element of \( \tilde{X}_{ij} \) (2.6). In particular, whether we are working with \( \tilde{Z} \) or \( \tilde{Z} \) we obtain the same Prym-translates of \( \tilde{X}_{ij} \) in \( A \)
(i.e., \( \tilde{X}_{ij} \) can also be regarded as the set of incident pairs of twisted cubics).

Inside \( D_Z \) we have the surfaces \( \Sigma(X_i) \), \( \Sigma((X_i)_\lambda) \) and (at least) 4 Prym-translates of \( \tilde{X}_{ij} \) because of the choices of ruling above \( p_i \) and \( p_j \). Recall (2.1) that the inverse images in \( \tilde{Z} \) of lines in \( \mathbb{P}^3 \) that are tangent to \( B \) are blown down to a point by \( AJ \) which is the base point for the embedding of \( D_Z \) in \( A \).

Taking as base point for the embedding of \( E_Z \) in \( A \) a line through \( p_i \) and \( p_j \) in \( \tilde{Z} \) we see that five of the curves in \( E_Z \) are also in \( D_Z \) (this is inspired by [C1]):

Let \( l_{ij} \) be the line in \( \mathbb{P}^3 \) through \( p_i, p_j \); \( l_{ij}', l_{ij}'' \) be the 2 lines in \( Z \) above it
and $R_i^1, R_j^2$ be the rulings of the exceptional quadrics in $\bar{Z}$ above $p_i$. If we choose $R_i^1+R_j^1$ as base point for the embedding of $E_Z$, the 5 curves are:

$$\bar{X}_i+R_i^1, (\bar{X}_iR^1+R_i^1, \bar{X}_j+R_j^2, (\bar{X}_jR^1+R_j^1, \bar{X}_{ij}-1_{ij}'' = \bar{X}_{ij}+R_i^2+R_j^2$$

viewed in $D_Z$.

Parametrizing $\Theta$ as in 2.4 with $(X_{12})_\lambda$ as parameter curve we can suppose that $E_Z = \Theta_a \Theta_b$ with $b-a = [p,q] \in \Sigma((X_{12})_\lambda)$. With the notations of 2.4 we have for instance $\bar{X}_{12} - 1_{12}'' = W_{p}+a$ (cf. 3.2), then, by 5.20, if $g_{12}$ is the $g_d^1$ relating $(X_{12})_\lambda$ to $X_1$ and $(X_2)_\lambda$, we have $h^0(g_{12} - \pi p - \pi q) > 0$. Letting $g_{12} = \pi p + \pi q + \pi r + \pi x$, we have by 2.5 that the intersections

$$\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]}, \Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,s]}$$

are unions of $W_{p}$ and Prym-translates of $\bar{X}_{1}$ and $(\bar{X}_{2})_\lambda$. Thus these four intersections give us the four possible combinations between 1 translate of $\bar{X}_{1}$ and 1 translate of $(\bar{X}_{2})_\lambda$ in $E_Z$, and we have for instance:

$$(\bar{X}_i+R_i^1) \cup ((\bar{X}_jR^1+R_j^1) \cup (\bar{X}_{12} - 1_{12}'' = \Theta_a \cap \Theta_{a+[p,q]} \cap \Theta_{a+[p,r]}$$

Letting $c = a+[p,r]$ we deduce:

$$\Theta_a \cap \Theta_b \cap \Theta_c \subset \Theta_a \cap \Theta_b \cap D_Z$$

d and a computation analogous to that of [BD1] (proof of proposition 2) gives an equation with constant coefficients:

$$(13.2) \quad \lambda \theta_a \theta_c + \mu \theta_a \theta_b + \nu \theta_c \theta_c + D_Z = 0$$

where $\theta_a$ is a nonzero section of $\Theta_A(\Theta_a)$ and we denote $D_Z$ and an equation for it by the same symbol (the coefficient of $D_Z$ is nonzero because $A$ is generic and its Kummer has no trisectants [ref]). From the above equation we deduce:

$$(13.3) \quad \Theta_a \cap \Theta_b \cap (\Theta_c \cup \Theta_{-c}) = \Theta_a \cap \Theta_b \cap D_Z$$

So $\Theta_a \cap \Theta_b \cap \Theta_{-c}$ contains $(\bar{X}_1)_\lambda+R_1^1$ and $\bar{X}_2+R_2^1$ and, by 3.2,

$$-c-a \in \Sigma(X_1) \cap \Sigma((X_2)_\lambda)$$

then we deduce from 5.13 that $-c-a = [s,t] \in \Sigma((X_{12})_\lambda)$ and also $h^0(h_{12} - \pi s - \pi t) > 0$.
where $h_{12}$ is the opposite $g_4^1$ of $g_{12}^1$. Hence the divisor $\Theta_{-c}$ contains a Prym-translate of $\tilde{X}_{12}$ which is also in $\Theta_a \cap \Theta_b \cap D_Z$ and can only be $\tilde{X}_{12} - l_{12}''$.

Now, as before, we also have: $h^0(h_{12} - \pi p - \pi q) > 0$. Putting $h_{12} = \pi p + \pi q + \pi r + \pi x'$, we obtain (as before):

$$\Theta_a \cap \Theta_b \cap (\Theta_d \cup \Theta_d) = \Theta_a \cap \Theta_b \cap D_Z$$

where for instance $d = a + [p, r]$ and

$$((\tilde{X}_1)_1 + R_1) \cup ((\tilde{X}_2)_2 + R_2) \cup ((\tilde{X}_{12} - l_{12}'')) = \Theta_a \cap \Theta_b \cap \Theta_d$$

Again the same type of computation as in [BD1] (proof of proposition 2) yields that two distinct translates of $\Theta$ (different from $\Theta$) have distinct traces on $\Theta_a \cap \Theta_b$ so our only possibility is $d = -c$. We thus obtain the following values for $a, b, c$:

$$a = 1/2([\sigma p, \sigma p] + [\sigma r, \sigma r'])$$
$$b = 1/2([q, q] + [\sigma r, \sigma r'])$$
$$c = 1/2(r, \sigma r')$$

with compatible halves.

This takes care of the black triangles in the octahedron (5.30).

(13.4) Our next aim is to write a relation between traces of translates of $\Theta$ on $E_Z$ and the incidence divisors $D_1$ on $E_Z$: for a line $l$ in $\tilde{Z}$, $D_1$ is the family of lines incident to it (see 2.1).

It is proven in [W1] (pp. 77-78) that if we fix a line $l_0$ in $\tilde{Z}$ then for all $l, D_1 - (\Theta_{l_0-l})|_{E_Z}$ is constant in Pic$E_Z$. Here $\Theta$ can be replaced by any translate of it, so replacing $\Theta$ by $\Theta_{d, d}$ ($d = a + [p, x]$ or $a + [p, \sigma x]$, see above) and taking $l = l_0 = l_{12} + R_1 + R_2$ we see that this constant is $(\tilde{X}_1 + R_1) - ((\tilde{X}_1)_1 + R_1)$ (the set theoretic computation of $D_1$ is easy, we need to see that the components occur with multiplicity 1.

For a double solid with smooth quartic branch locus we deduce from [C1] and [W1] that the homology class of $D_1 \subset E_Z$ in $JZ$ is $1/3(2\Theta^8/g_1) \Theta = 6\Theta^9/g_9!$. Degenerating one double point at a time to the case of a branch locus with six double points (as in 2.1) we see that the homology class of $D_1$ in $A = J\tilde{Z}$ is $\Theta^3$. So the constant $D_1 - (\Theta_{l_0-l})|_{E_Z}$ is algebraically equivalent to 0 in $A$.)
The white triangles in the octahedron are incidence divisors on $E_Z$.

Now by the Seesaw theorem ([M1] p. 54) we can complete the result of [W1] in this case (p. 77):

If $\phi : E_Z \times E_Z \to A$ is the sum and $\rho_i$ the projections $E_Z \times E_Z \to E_Z$, then:

$$\phi^*\mathcal{O}_A(\Theta_d) \equiv \mathfrak{d} \otimes \rho_1^*\mathcal{O}(\tilde{X}_1 - (\tilde{X}_1)_\lambda) \otimes \rho_2^*\mathcal{O}(\tilde{X}_1 - (\tilde{X}_1)_\lambda)$$

where $\mathfrak{d}$ is the incidence divisor on $E_Z \times E_Z$ and we omit the base points to simplify notations.
BIBLIOGRAPHY


