Exercise 7.1. If $A$ and $C$ are nonsingular matrices, show that the matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is nonsingular.

Exercise 7.2. Write down the optimality conditions for each of the following linear programs in complementary slackness form.

(a) minimize $c^T x$ subject to $Ax \geq b$
(b) minimize $c^T x$ subject to $Ax = b, Dx \geq f$
(c) minimize $c^T x$ subject to $Ax = b, x \geq 0$

Exercise 7.3. Consider the standard-form problem of minimizing $c^T x$ subject to $Ax = b, x \geq 0$, with

$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -3 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $c = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$.

(a) Use any method of your choice to find a vertex for this constraint set.
(b) Is the vertex optimal? Explain why or why not.

Exercise 7.4. Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 - 2x_3 - 2x_4 \\
\text{subject to} & \quad x_2 + 2x_3 \leq 1 \\
& \quad x_1 + x_2 + x_3 + x_4 \leq 1 \\
& \quad x_1 - x_2 - 5x_3 + 3x_4 \leq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_3 \geq 0 \\
& \quad x_4 \geq 0.
\end{align*}
\]

(a) Convert this problem into standard-form min $c^T x$ subject to $Ax = b, x \geq 0$.
(b) Compute one iteration of the standard-form simplex method for this problem, starting at the basic solution $x_B = (\frac{1}{2}, \frac{1}{2}, 2)^T$ defined by columns 3, 4 and 7 of $A$. Show your work. Be sure to write down the objective function, basic set, nonbasic set, $\pi$-vector and reduced costs at both the beginning and end of the iteration. Check that the new iterate is feasible, with improved objective value.

Exercise 7.5. Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad d^T w \\
\text{subject to} & \quad Gw \geq f, \quad w \geq 0,
\end{align*}
\]

where $G$ is an $m \times n$ matrix.
(a) Convert this problem to all-inequality form and to standard form. In both cases, write down the problem in terms of “x” for the vector of variables, “c” for the objective, and “A” for the matrix of constraints.

(b) Formulate the dual form of the linear program.

(c) Suppose that you are given a specific problem in the above LP form. Briefly discuss the circumstances under which you would use the primal or dual form to solve the problem.

Exercise 7.6. Formulate dual problems for the following linear programs:

(a)
\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax \geq b, \quad a^T x = \beta.
\end{align*}
\]

(b)
\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax = b, \quad Bx \leq d, \quad x \geq 0.
\end{align*}
\]