Math 171A: Linear Programming

Lecture 1

Overview of the Class: Introduction to Optimization

Elizabeth Wong  © 2019

http://math.ucsd.edu/~elwong/171a

Monday, January 7th, 2019
# Overview of Math 171A

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Office Hours</th>
<th>Thursday Section(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elizabeth Wong</td>
<td>Mondays, Fridays 2p - 3:15p APM 5848</td>
<td></td>
</tr>
<tr>
<td>Xindong Tang</td>
<td>APM 6303</td>
<td>A01, A02 (8pm, 9pm in B402A)</td>
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<tr>
<td>Ziyun Zhu</td>
<td>APM 2202</td>
<td>A03, A04 (8pm, 9pm, in 2402)</td>
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<tr>
<td>Bingni Guo</td>
<td>APM 6442</td>
<td>A05 (7pm in B402A)</td>
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</tbody>
</table>

No discussions this week!

Class webpage: http://math.ucsd.edu/~elwong/171a
Overview of Math 171A

- Lecture slides are available for download on the class website
- Generally will be available the evening before lecture
- Will be updated after class
- If you print, please be mindful of the paper used – there are paper-friendly 4x4 versions of the slides available
- The login is your UCSD username and the password is your UCSD PID
- For example, 'elwong' and 'A01234567'
Overview of Math 171A

The grade is based on homework, 2 midterms and a final

Homework                          20%
Midterm #1  (Friday Feb 1):         20%
Midterm #2  (Friday Mar 1):         20%
Final     (Friday, March 22, 11:30am–2:30pm): 40%
Overview of Math 171A

Homework

- Written part: submitted through www.gradescope.com
- Programming part: Jupyter notebooks on datahub.ucsd.edu (Julia!)
  - Some programming experience recommended (Matlab, Math 18/20D)
  - No experience with Julia/Jupyter required

This week:
- Expect an email from gradescope.com
- Check that you can login to datahub.ucsd.edu
What is Optimization?

*Optimization* is the search for the best solution to a problem under a given set of circumstances.

The *study* of optimization involves the formulation of a *mathematical model* of a given process or resource for which “optimizing” means minimizing or maximizing a function.
What is Optimization?

Optimization is the search for the best solution to a problem under a given set of circumstances.

The study of optimization involves the formulation of a mathematical model of a given process or resource for which “optimizing” means minimizing or maximizing a function.
What is Optimization?

1. Model the problem
2. Optimize the model

Math 171A focuses on Step (2), in particular, on algorithms for optimization.
Topics for this quarter

• The formulation of simple linear programs
  • The graphical method

• Review of linear algebra

• Linear programs with constraints in all-inequality form and standard form
  • Optimality conditions

• The simplex method

• Finding a feasible point

• Primal and dual forms of a linear program

• Interior methods
A Simple Modeling Example

The JuiceCo company produces CranApple and AppleBerry juice. Both are made from a blend of apple and cranberry syrup. At the moment, JuiceCo has 200 quarts of cranberry juice and 100 quarts of apple juice. CranApple requires 1 quart of apple and 3 quarts of cranberry juice. AppleBerry requires 2 quarts of apple and 2 quarts of cranberry. CranApple sells for $3 per quart and AppleBerry for $4. How many quarts of CranApple and AppleBerry juice should JuiceCo produce to maximize profits?
Step 1: Identify the variables

\[ x = \text{quarts of cranapple juice produced} \]
\[ y = \text{quarts of appleberry juice produced} \]

The problem has 2 variables. We want to find the “best” or “optimal” values for \( x \) and \( y \).
Variables

**Step 1:** Identify the *variables*

\[ x = \text{quarts of cranapple juice produced} \]
\[ y = \text{quarts of appleberry juice produced} \]

The problem has 2 *variables*. We want to find the “best” or “optimal” values for \( x \) and \( y \).
Step 1: Identify the variables

\( x = \) quarts of cranapple juice produced
\( y = \) quarts of appleberry juice produced

The problem has 2 variables. We want to find the “best” or “optimal” values for \( x \) and \( y \)
Step 2: Identify the objective

Find $x$ and $y$ to maximize profit

*CranApple sells for $3 per quart and AppleBerry for $4.*

The *objective function* is $3x + 4y$
Objective

Step 2: Identify the objective

Find $x$ and $y$ to maximize profit

*CranApple sells for $3 per quart and AppleBerry for $4.*

The objective function is $3x + 4y$
Step 2: Identify the objective

Find $x$ and $y$ to maximize profit

CranApple sells for $3$ per quart and AppleBerry for $4$.

The objective function is $3x + 4y$
Step 3: Identify the constraints

What are some of the limitations on JuiceCo during production?

- JuiceCo has 200 quarts of cranberry juice and 100 quarts of apple juice.
- CranApple requires 1 quart of apple and 3 quarts of cranberry juice.
- AppleBerry requires 2 quarts of apple and 2 quarts of cranberry.

\[
x + 2y \leq 100 \quad \text{(total apple juice available)}
\]
\[
3x + 2y \leq 200 \quad \text{(total cranberry juice available)}
\]

Additionally, we can’t produce negative amounts of cranapple or appleberry juice! \(x, y \geq 0\)
Step 3: Identify the constraints

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Constraints

**Step 3: Identify the constraints**

What are some of the limitations on JuiceCo during production?

- *JuiceCo has 200 quarts of cranberry juice and 100 quarts of apple juice.*

- *CranApple requires 1 quart of apple and 3 quarts of cranberry juice.*

- *AppleBerry requires 2 quarts of apple and 2 quarts of cranberry.*

\[
\begin{align*}
  x + 2y & \leq 100 \quad \text{(total apple juice available)} \\
  3x + 2y & \leq 200 \quad \text{(total cranberry juice available)}
\end{align*}
\]

Additionally, we can’t produce negative amounts of cranapple or appleberry juice!  \[ \Rightarrow x, y \geq 0 \]
Mathematical problem

maximize \[ 3x + 4y \]

subject to \[ x + 2y \leq 100 \]
\[ 3x + 2y \leq 200 \]
\[ x \geq 0 \]
\[ y \geq 0 \]

Notice, the objective function and the constraints are all linear.
Mathematical problem

maximize \limits_{x,y} 3x + 4y
subject to \quad x + 2y \leq 100
3x + 2y \leq 200
x \geq 0
y \geq 0

Notice, the objective function and the constraints are all *linear*
Structure of the class

Math 171A: minimizing \textit{linear} functions subject to \textit{linear} constraints

\textit{Linear programming}

Math 171B: minimizing \textit{nonlinear} functions subject to \textit{nonlinear} constraints

\textit{Nonlinear programming}
Some examples of optimization problems

Data science:
  • Compressive sensing: minimizing combinations of norms.
  • Primal and dual forms of linear support-vector machines.

Inverse problems:
  • Medical imaging
  • Geophysical applications
    • Estimating the Earth’s electrical conductivity
    • Mineral and oil exploration
  • Controlling the environment
    • Cleaning groundwater contaminants
Air travel:

- Flight attendants
  - A flight scheduling algorithm assigned the flight crew to your plane

- Aircraft design
  - An optimization algorithm was used to design shape of the plane

- Passenger routing
  - NBA game locations and dates are determined using optimization
Design technology:

- **Yacht design**
  - Optimal design of America’s Cup boats

- **Space-X Falcon 9**
  - The take-off and re-entry trajectories calculated to minimize structural stress and heating
  - Determination of the surface temperature using sensors
  - Minimization of fuel consumption and maximization of payload

- **Calculation of trajectories for spacecraft**
  - Optimal control of long-term interplanetary missions
  - Design of orbiters

- **Control of unmanned autonomous vehicles (UAVs)**
Can we use methods from calculus?

For example:

\[
\begin{align*}
\text{minimize} & \quad x^2 \\
\text{The objective function } f(x) = x^2 & \text{ is minimized at } x^* = 0 \\
\text{The derivative is zero at } x^* & \\
\frac{df}{dx} = f'(x) &= 2x = 0
\end{align*}
\]
Similarly,

$$\minimize_{x,y} \quad x^2 + y^2$$
Examples from calculus

The function $f(x) = x^2 + y^2$ is minimized at $x^* = 0$, $y^* = 0$.

The partial derivatives are zero at $x^*$ and $y^*$

\[
\frac{\partial f}{\partial x} = 2x = 0 \\
\frac{\partial f}{\partial y} = 2y = 0
\]
Our problem is more difficult because of the inequality constraints.

How do we solve this problem?

maximize \( 3x + 4y \)
subject to \( x + 2y \leq 100 \)
\( 3x + 2y \leq 200 \)
\( x \geq 0 \)
\( y \geq 0 \)
Problem Geometry

maximize $3x + 4y$
subject to $3x + 2y \leq 200$
$x + 2y \leq 100$
$x \geq 0$
y $\geq 0$
Problem Geometry

maximize $3x + 4y$
subject to $3x + 2y \leq 200$
$x + 2y \leq 100$
$x \geq 0$
$y \geq 0$
Problem Geometry

maximize \( x, y \)

subject to
\[
3x + 2y \leq 200
\]
\[
x + 2y \leq 100
\]
\[
x \geq 0
\]
\[
y \geq 0
\]
Problem Geometry

maximize $3x + 4y$

subject to $3x + 2y \leq 200$

$x \geq 0$

$y \geq 0$
maximize $3x + 4y$

subject to $3x + 2y \leq 200$
$x + 2y \leq 100$
$x \geq 0$
$y \geq 0$
Basic result of linear programming

The *optimal solution* \((x^*, y^*)\) lies at a “corner-point” of the boundary of the feasible region.