Lecture 10

Feasible Directions for Inequality Constraints

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Wednesday, January 30, 2019
Recap

Optimality conditions for ELP

Linear programming with equality constraints:

ELP \[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

A point \( \bar{x} \) is optimal if and only if

\[
A\bar{x} = b \quad \text{and there is a vector} \quad \lambda \quad \text{satisfying} \quad A^T\lambda = c
\]

The components of \( \lambda \) are called the \textit{Lagrange multipliers}.
General LP

LP minimize \( c^T x \)
subject to \( A x \geq b \)

What are the \textit{feasible directions} for this problem?
Result 1

If $a_i^T x \geq b_i$ is *inactive* at $\bar{x}$, then *every* nonzero $p$ is a feasible direction at $\bar{x}$. 

\[ a_i^T x > b_i \quad a_i^T x = b_i \quad a_i^T x < b_i \]
Result 2

If $a_i^T x \geq b_i$ is active at $\bar{x}$, then $p$ is a feasible direction only if $a_i^T p \geq 0$. 

\[ a_i^T x > b_i \quad a_i^T x < b_i \quad a_i^T x = b_i \]
Proofs: We have

$$r_i(\bar{x} + \alpha p) = r_i(\bar{x}) + \alpha a_i^T p$$

For Result 1, $a_i^T x \geq b_i$ is inactive at $\bar{x}$ and we have $r_i(\bar{x}) > 0$. Thus,

$$r_i(\bar{x} + \alpha p) = r_i(\bar{x}) + \alpha a_i^T p \geq 0$$

$\Rightarrow r_i(\bar{x} + \alpha p) > 0$ for $\alpha$ sufficiently small, regardless of the sign of $a_i^T p$. 
For Result 2, $a_i^T x \geq b_i$ is active at $\bar{x}$. We have $r_i(\bar{x}) = 0$, and

$$r_i(\bar{x} + \alpha p) = r_i(\bar{x}) + \alpha a_i^T p = \alpha a_i^T p$$

As $\alpha$ can be any nonnegative step, it follows that:

$r_i(\bar{x} + \alpha p) \geq 0$ only if $a_i^T p \geq 0$. $\blacksquare$
A geometric view...

Recall, if $\theta$ is the angle between $a_i$ and $p$

$$a_i^T p = \|a_i\|\|p\| \cos \theta$$

$p$ is a feasible direction if

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$
The *set of feasible directions* for $Ax \geq b$ is

$$\{p : p \neq 0, \ a_i^T p \geq 0 \text{ for all active constraint indices } i\}.$$
The set of feasible directions for $Ax \geq b$ is

$$\{p : p \neq 0, a_i^T p \geq 0 \text{ for all active constraint indices } i\}.$$ 

This condition can be written in matrix-form if we gather our active constraints together.

**Definition**

The active set at $x$ is the set of row indices

$$\mathcal{A}(x) = \{i : a_i^T x = b_i\}.$$
Given an active set $\mathcal{A}(x)$, we define the \textit{active-constraint matrix} $A_a(x) \triangleq \text{matrix of active constraint normals}$

$A_a(x)$ is made up of a subset of rows of $A$. We denote the size of $\mathcal{A}(x)$ (and equivalently the number of rows in $A_a$ as $m_a(x)$.
Similarly,

\[ b_a(x) \triangleq \text{vector of right-hand sides for the active constraints} \]

If the point \( x \) where these quantities are defined is obvious, we may drop the dependence on \( x \) for simplicity.

Any feasible point \( \bar{x} \) satisfies

\[ A_a\bar{x} = b_a \quad \text{and} \quad A\bar{x} \geq b. \]
The set of feasible directions for \( Ax \geq b \) is

\[
\{ p : p \neq 0, \ a_i^T p \geq 0 \ \text{for all active constraint indices } i \}\).
\]

In active-set terminology, the set of feasible directions at \( \bar{x} \) is

\[
\{ p : p \neq 0, \ a_i^T p \geq 0 \ \text{for } i \in \mathcal{A}(\bar{x}) \}\).
\]

or equivalently,

\[
\{ p : p \neq 0, \ A_a(\bar{x})p \geq 0 \}.
\]
Step length for feasible directions

\[ A(\bar{x} + \alpha p) \geq b \text{ for all } 0 < \alpha \leq \sigma \]

\( \sigma \) is the \textit{largest step} we can take along a feasible direction and still remain feasible.

There may be a finite \( \sigma \) associated with a feasible direction.
We want to compute the *step to a constraint hyperplane* along a direction $p$. 
Step to a constraint hyperplane

Let \( \bar{x} \) be any point in \( \mathbb{R}^n \) and \( p \) be any nonzero direction. We compute the step to the \( i \)th constraint hyperplane \( a_i^T x \geq b_i \).

Moving along \( p \) from \( \bar{x} \) changes the residual:

\[
 r_i(\bar{x} + \alpha p) = r_i(\bar{x}) + \alpha a_i^T p
\]

If \( a_i^T p \neq 0 \), the constraint \( a_i^T x \geq b_i \) becomes active at \( \alpha = \sigma_i \) when

\[
 0 = r_i(\bar{x} + \sigma_i p) = r_i(\bar{x}) + \sigma_i a_i^T p
\]
0 = r_i(\bar{x} + \sigma_i p) = r_i(\bar{x}) + \sigma_i a_i^T p

Rearranging the equation gives

\[ \sigma_i = -\frac{r_i(\bar{x})}{a_i^T p} = \frac{r_i(\bar{x})}{-a_i^T p} \]
Definition

The *step to the constraint* $a_i^T x \geq b_i$ from $\bar{x}$ along a nonzero direction $p$ is

$$\sigma_i = \frac{r_i(\bar{x})}{-a_i^T p} \text{ if } a_i^T p \neq 0.$$

A *positive* $\sigma$ means the constraint is in front of us

A *negative* $\sigma$ means the constraint is behind us
We have to be careful how we define $\sigma_i$ when $a_i^T p = 0$.

Suppose that $\bar{x}$ is feasible for $a_i^T x \geq b_i$ (i.e., $r_i(\bar{x}) \geq 0$).

**Definition**

The step to the constraint $a_i^T x \geq b_i$ from a feasible point $\bar{x}$ along a nonzero $p$ is given by:

$$
\sigma_i = \begin{cases} 
\frac{r_i(\bar{x})}{-a_i^T p} & \text{if } a_i^T p \neq 0 \\
+\infty & \text{if } a_i^T p = 0 \text{ and } r_i(\bar{x}) > 0 \\
\text{undefined} & \text{if } a_i^T p = 0 \text{ and } r_i(\bar{x}) = 0
\end{cases}
$$
Example:

\[
\begin{align*}
 x_1 + x_2 & \geq 4 \\
 x_1 & \geq -1 \\
 6x_1 - x_2 & \leq 18 \\
 x_1 + 2x_2 & \geq 6 \\
 3 & \leq x_2 \leq 6 \quad \text{(i.e., } x_2 \geq 3 \text{ and } x_2 \leq 6) \end{align*}
\]

Constraint #1: \( x_1 + x_2 \geq 4 \), \( a_1^T = (1 \ 1) \), \( b_1 = 1 \)

Constraint #2: \( x_1 \geq -1 \), \( a_2^T = (1 \ 0) \), \( b_2 = -1 \)

Constraint #3: \( -6x_1 + x_2 \geq -18 \), \( a_3^T = (-6 \ 1) \), \( b_3 = -18 \)

Constraint #4: \( x_1 + 2x_2 \geq 6 \), \( a_4^T = (1 \ 2) \), \( b_4 = 6 \)

Constraint #5: \( x_2 \geq 3 \), \( a_5^T = (0 \ 1) \), \( b_5 = 3 \)

Constraint #6: \( -x_2 \geq -6 \), \( a_6^T = (0 \ -1) \), \( b_6 = -6 \)
This is just $Ax \geq b$, with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -6 & 1 \\ 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 4 \\ -1 \\ -18 \\ 6 \\ 3 \\ -6 \end{pmatrix}$$
Consider the point $\bar{x}$ and direction $p$:

$$\bar{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$
\[ r(\bar{x}) = A\bar{x} - b = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \\ 3 \\ 0 \\ 3 \end{pmatrix} \leftarrow \text{constraint \#5 is active at } \bar{x} \]

\[ \Rightarrow \bar{x} \text{ is feasible, with } A(\bar{x}) = \{5\}, \ A_a = (0 \quad 1) \text{ and } b_a = (3). \]
\[ Ap = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -6 & 1 \\ 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 6 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{← constraint #5} \]

\[ A_a p = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0 \geq 0 \]

\[ \Rightarrow p \text{ is a feasible direction.} \]
At $\bar{x}$ we compute the step to each of the constraint hyperplanes...

$$r(\bar{x}) = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \quad A p = \begin{pmatrix} -1 \\ -1 \\ 6 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Step to constraint #1:

$$\sigma_1 = \frac{r_1(\bar{x})}{(-a_1^T p)} = \frac{2}{(-(-1))} = 2$$
\[ r(\bar{x}) = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \quad Ap = \begin{pmatrix} -1 \\ -1 \\ 6 \\ -1 \\ 0 \\ 0 \end{pmatrix} \]

Step to constraint \#2:

\[ \sigma_2 = \frac{r_2(\bar{x})}{(-a_2^T p)} = \frac{4}{((-(-1)))} = 4 \]
Step to constraint #3:

\[
\sigma_3 = \frac{r_3(\bar{x})}{-a_3^T p} = \frac{3}{-6} = -\frac{1}{2}
\]

⇒ constraint #3 is “behind us” for positive steps along p.
\[
\begin{align*}
r(\bar{x}) &= \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \quad Ap &= \begin{pmatrix} -1 \\ -1 \\ 6 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\
\text{Step to constraint \#4:} \quad \sigma_4 &= \frac{r_4(\bar{x})}{-a_T^T p} = \frac{3}{(-(-1))} = 3
\end{align*}
\]
\[ r(\bar{x}) = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \\ 0 \end{pmatrix}, \quad Ap = \begin{pmatrix} -1 \\ -1 \\ 6 \\ -1 \\ 0 \end{pmatrix} \]

Step to constraint \#5:

\[ \sigma_5 = \frac{r_5(\bar{x})}{-a_5^T \bar{p}} = \text{undefined} \]
\[
\begin{pmatrix}
2 \\
4 \\
3 \\
3 \\
0 \\
3
\end{pmatrix}, \quad
\begin{pmatrix}
-1 \\
-1 \\
6 \\
-1 \\
0 \\
0
\end{pmatrix}
\]

Step to constraint \#6:

\[
\sigma_6 = \frac{r_6(\bar{x})}{(-a_6^T p)} = \frac{3}{(-0)} = +\infty
\]

because \( p \) is orthogonal to \( a_6 \) and can never intersect the hyperplane.