Recap: Feasible directions for inequalities

Definition
The set of feasible directions for $Ax \geq b$ at a point $\bar{x}$ is
\[ \{ p : p \neq 0, A_\delta(\bar{x})p \geq 0 \}, \]
where $A_\delta(\bar{x})$ is the active-constraint matrix for the active set $A(\bar{x})$.

Definition
The step to the constraint $a_i^T x \geq b_i$ from a feasible point $\bar{x}$ along a nonzero $p$ is given by:
\[
\sigma_i = \begin{cases} 
  \frac{r_i(\bar{x})}{-a_i^T p} & \text{if } a_i^T p \neq 0 \\
  +\infty & \text{if } a_i^T p = 0 \text{ and } r_i(\bar{x}) > 0 \\
  \text{undefined} & \text{if } a_i^T p = 0 \text{ and } r_i(\bar{x}) = 0
\end{cases}
\]

If $\bar{x}$ is feasible, $\bar{x} + \alpha p$ can violate a constraint only if $p$ points towards its infeasible half space.
\( \bar{x} + \alpha p \) can violate \( a_i^T x \geq b_i \) only if \( a_i^T p < 0 \)

**Definition (Decreasing constraints)**

Given a feasible \( \bar{x} \) and a nonzero \( p \), the set of **decreasing constraints** is the set of constraint indices

\[ D = \{ i : a_i^T p < 0 \} \]

\( \bar{x} + \alpha p \) can **never** violate \( a_i^T x \geq b_i \) if \( a_i^T p \geq 0 \).

**Definition (Non-decreasing constraints)**

Given a feasible \( \bar{x} \) and a nonzero \( p \), the set of **non-decreasing constraints** is the set of constraint indices

\[ I = \{ i : a_i^T p \geq 0 \} \]

From the example in our last lecture, \( Ax \geq b \), with

\[
A = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
-6 & 1 \\
1 & 2 \\
0 & 1 \\
0 & -1
\end{pmatrix}
\quad \text{and} \quad
b = \begin{pmatrix}
4 \\
-1 \\
-18 \\
6 \\
3 \\
-6
\end{pmatrix}
\]

We have

\[
r(\bar{x}) = \begin{pmatrix}
2 \\
4 \\
3 \\
3 \\
0 \\
0
\end{pmatrix}
\quad \text{and} \quad
Ap = \begin{pmatrix}
-1 \\
-1 \\
6 \\
-1 \\
0 \\
0
\end{pmatrix}
\]

\[ D = \{ 1, 2, 4 \} \quad \text{and} \quad I = \{ 3, 5, 6 \} \]

\[ \sigma = \min_{i \in D} \sigma_i = \min\{ \sigma_1, \sigma_2, \sigma_4 \} = \min\{ 2, 4, 3 \} = 2 \]

\( \Rightarrow \) the step \( \sigma = 2 \) is the step to the boundary of constraint \#1.

\( \Rightarrow \) Constraint \#1 is “blocking” our step along the direction \( p \).
Definition (Blocking constraint)
Given the largest step $\sigma$ from a feasible $\bar{x}$ along a feasible direction $p$, then any constraint such that
\[ r_t(\bar{x} + \sigma p) = 0 \]
is known as a blocking constraint.

The step $\sigma_t$ to a blocking constraint satisfies
\[ \sigma_t = \sigma = \min_{i \in D} \sigma_i \]
The step $\sigma$ is unique, but there may be many blocking constraints.

Summary
If $p$ is a feasible direction, then
\[ r(\bar{x} + \alpha p) \geq 0 \quad \text{for all} \quad 0 < \alpha \leq \sigma \]

- The active constraints at $\bar{x}$ determine the set of feasible directions $p$.
- The inactive constraints at $\bar{x}$ determine the definition of $\sigma$. 
First, we show that the feasible region has a nice property called convexity.

A little background . . .

**Definition (convex set)**

A set $S$ is **convex** if, for every $x, y \in S$, it holds that

$$z = (1 - \theta)x + \theta y \in S$$

for all $0 \leq \theta \leq 1$. 

Note that

$$z = (1 - \theta)x + \theta y$$
$$= x + \theta(y - x)$$
$$= x + \theta p, \text{ with } p = y - x$$

i.e., steps along any $p$ joining $x$ and $y \in S$ give a point in $S$. 

The feasible region $\mathcal{F} = \{x : Ax \geq b\}$ is either empty or convex.

**Proof:** The result is trivial if $\mathcal{F}$ is empty (i.e., no feasible point).
Assume that $\mathcal{F}$ is not empty, with $x, y \in \mathcal{F}$, i.e.,

$$x \in \mathcal{F} \implies Ax \geq b$$
$$y \in \mathcal{F} \implies Ay \geq b$$

If $\theta \in [0, 1]$ then

$$A((1-\theta)x + \theta y) = (1-\theta)Ax + \theta Ay$$
$$\geq (1-\theta)b + \theta b = b$$

Thus, $(1-\theta)x + \theta y \in \mathcal{F}$.  

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**Result**

$x^*$ is a minimizer of $c^Tx$ subject to $Ax \geq b$ if and only if

- $Ax^* \geq b$
- $c^T(x^* + \alpha p) \geq c^Tx^*$ for all feasible directions $p$ at $x^*$, and all $\alpha \geq 0$ sufficiently small.

The second condition is equivalent to requiring that

$$c^Tp \geq 0$$

for all feasible directions $p$ at $x^*$

i.e., all $p$ such that $A_p \geq 0$.

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**Implication:**

Given $x^* \in \mathcal{F}$, we can write every feasible $x$ in the form

$$x = x^* + p, \quad \text{with} \quad p = x - x^*$$

Then, if $A_p = A_p(x^*)$, it must hold that

$$A_p = A_p(x^*) = A_p x - A_p x^* \geq b - b = 0$$

$$\implies A_p \geq 0,$$

and so $p$ is a feasible direction at $x^*$.

The convexity of $\mathcal{F}$ implies that every feasible $x$ is reachable from $x^*$ by taking a step along some feasible direction.

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**Result**

$x^*$ is a minimizer of $c^Tx$ subject to $Ax \geq b$ if and only if

- $Ax^* \geq b$
- $c^Tp \geq 0$ for all feasible directions $p$ at $x^*$ (i.e., $A_p \geq 0$)

**Result**

$x^*$ is a minimizer of $c^Tx$ subject to $Ax \geq b$ if and only if

- $Ax^* \geq b$
- There is no direction $p$ such that
  $$c^Tp < 0 \quad \text{and} \quad A_p \geq 0$$

In both cases, $A_p = A_p(x^*)$. 
**Definition**

A direction $p$ such that $c^Tp < 0$ and $A_a p \geq 0$ is known as a *feasible descent direction*.

ELP: $c^Tp \geq 0$ for all $Ap = 0$

LP: $c^Tp \geq 0$ for all $A_a p \geq 0$

Both conditions imply the existence of *Lagrange multipliers*.

**Theorem**

Consider minimizing $c^Tx$ subject to $Ax \geq b$.

(A) If no points satisfy $Ax \geq b$, the LP has no solution;

(B) If there exists a point $x^*$ satisfying the three conditions

$$Ax^* \geq b, \quad c = A_a^T \lambda_a^*, \quad \text{and} \quad \lambda_a^* \geq 0,$$

where $A_a$ is the active constraint matrix at $x^*$, then $\ell^* = c^Tx^*$ is the minimum value of $c^Tx$ in the feasible region, and $x^*$ is a minimizer;

(C) If the constraints $Ax \geq b$ are feasible, then $\ell(x)$ is unbounded below in the feasible region if and only if the two conditions $c = A_a^T \lambda_a^*$ and $\lambda_a^* \geq 0$, are not satisfied at any feasible point.

**Result (Farkas’ Lemma)**

(A) $c^Tp \geq 0$ for all $p$ such that $A_a p \geq 0$

if and only if

(B) $c = A_a^T \lambda_a^*$ for some $\lambda_a^* \geq 0$

**Result**

A vector $p$ exists such that

(C) $c^Tp < 0$ and $A_a p \geq 0$

if and only if

(D) $c$ cannot be written as $A_a^T \lambda_a^*$ for some $\lambda_a^* \geq 0$

**Vertices**
Definition (Dependent constraints)

A constraint $a^T x \geq \beta$ is said to be dependent on the constraints $a_i^T x \geq b_i$, $i = 1, 2, \ldots, r$, if the normal vector $a$ is linearly dependent on the normal vectors $a_1, a_2, \ldots, a_r$.

If we define

$$A_r = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_r^T \end{pmatrix}$$

and $a^T x \geq \beta$ is dependent on $a_i^T x \geq b_i$, $i = 1, 2, \ldots, r$, then

$$a = \sum_{i=1}^{r} a_i w_i \quad \text{for some} \quad \{w_i\} \text{ not all zero}$$

i.e.,

$$a = A_r^T w \quad \text{for} \quad w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix} \neq 0$$

Definition (Vertex)

Given a set of $m$ linear constraints in $n$ variables, a vertex is a feasible point at which there are at least $n$ linearly independent constraints active. (Equivalently, $A_a$ has at least one subset of $n$ linearly independent rows, which implies that $A_a$ has rank $n$.)

- At a vertex it is necessary that $m_a \geq n$
- The condition $m_a \geq n$ is not sufficient for $\bar{x}$ to be a vertex, e.g.,

$$x_1 + 2x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 2$$
$$\frac{1}{2}x_1 + x_2 \geq \frac{1}{2}$$

At the point

$$\bar{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \text{ then } A_a = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ \frac{1}{2} & 1 \end{pmatrix}, \text{ with } \text{rank}(A_a) = 1$$