Recap

A *vertex* is a *feasible* point at which at least $n$ linearly independent constraints are active.

The active constraint matrix $A_a$ has rank $n$ at a vertex.
Example:

\[ \begin{align*}
    x_1 + x_2 & \geq 1 \\
    x_1 & \geq 0 \\
    -x_1 & \geq -2 \\
    x_2 & \geq 0 \\
    x_1 + 2x_2 & \geq 1
\end{align*} \]

In matrix-vector form \( Ax \geq b \), with

\[
A = \begin{pmatrix}
  1 & 1 \\
  1 & 0 \\
  -1 & 0 \\
  0 & 1 \\
  1 & 2
\end{pmatrix}
\]

and

\[
b = \begin{pmatrix}
  1 \\
  0 \\
  -2 \\
  0 \\
  1
\end{pmatrix}
\]
\[
A = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
-1 & 0 \\
0 & 1 \\
1 & 2
\end{pmatrix}
\]

\[
A_a = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix} \text{(vertex)}
\]

\[
A_a = \begin{pmatrix}
1 & 1 \\
1 & 2
\end{pmatrix} \text{(vertex)}
\]

\[
A_a = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \text{(non-vertex)}
\]
### Definition (Nondegenerate vertex)

A vertex at which exactly $n$ constraints are active is called a **nondegenerate vertex**.

### Result

If $\bar{x}$ is a nondegenerate vertex, then $A_a$ is nonsingular.

### Definition (Degenerate vertex)

A vertex at which more that $n$ constraints are active is called a **degenerate vertex**.
\( A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \)

\( A_a = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) (nondegenerate vertex)

\( A_a = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \) (degenerate vertex)
Result

At a degenerate vertex \( \bar{x} \), every subset of \( n \) linearly independent constraints \textit{uniquely} defines \( \bar{x} \).

Recall, tall thin systems of equations (with full-rank) have \textit{unique} solutions.
When can we *guarantee* that $Ax \geq b$ has a vertex?

How do we *find* a vertex?

**Result (Existence of a vertex)**

If the following conditions hold:

1. $F = \{x : Ax \geq b\}$ has at least one point.

2. $\text{rank}(A) = n$

then $F$ has at least one vertex.
Result (Existence of a vertex)

If the following conditions hold:

1. $F = \{x : Ax \geq b\}$ has at least one point.

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then $F$ has at least one vertex.
(Constructive) Proof:

The assumptions are:

- There is at least one feasible point, say $x_0$ such that $Ax_0 \geq b$.
- There is at least one subset of $n$ independent rows of $A$. 
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- There is at least one feasible point, say $x_0$ such that $Ax_0 \geq b$.

- There is at least one subset of $n$ independent rows of $A$.

Consider any feasible point $x_0$. Define the active set at $x_0$, i.e.,

$$A_0x_0 = b_0$$

Assume that $\text{rank}(A_0) < n$ (otherwise there is nothing to prove!)

$A_0$ may be any shape (because it may have dependent rows).
We construct a *sequence of feasible points*, labeled as

\[ x_0, \ x_1, \ \ldots, \ x_k, \ x_{k+1}, \ \ldots \]

with active-set matrices

\[ A_0, \ A_1, \ \ldots, \ A_k, \ A_{k+1}, \ \ldots \]

such that

\[ \text{rank}(A_{k+1}) > \text{rank}(A_k) \]

...we stop when we get to a point with an active-constraint matrix that has rank \( n \) (i.e., when we get to a vertex).

Now, we show how to get \( x_{k+1} \) from \( x_k \) (i.e., the \( k \)th iteration)
$k$th iteration: Step 1:

The previous steps give $x_k$, $A_k$ and $b_k$ such that

$$A_k x_k = b_k \quad \text{and} \quad \text{rank}(A_k) < n$$

First, we identify an inactive constraint $a_j^T x \geq b_j$ that is independent of the active constraints, i.e.,

$$\text{rank} \left( \begin{pmatrix} A_k \\ a_j^T \end{pmatrix} \right) = \text{rank}(A_k) + 1$$

It is always possible to find $a_j^T x \geq b_j$ because $A$ has at least one subset of $n$ linearly independent constraints.

Note that $r_j(x_k) > 0$ because $a_j^T x \geq b_j$ is inactive.
kth iteration: Step 2:

Compute a feasible direction $p_k$ such that:

(A) any positive step along $p_k$ reduces the residual $r_j(x)$

i.e., “$p_k$ points towards the infeasible side of $a_j^T x \geq b_j$”

(B) the active-constraints stay active for any step along $p_k$

i.e., the active-constraint residuals stay fixed at zero.

Conditions (A) and (B) $\Rightarrow$ $p_k$ satisfies

(A) $a_j^T p_k < 0$

(B) $a_i^T p_k = 0$ for all $i \in A(x_k)$
Conditions (A) and (B) $\Rightarrow p_k$ satisfies:

(A) $a_j^T p_k < 0$

(B) $a_i^T p_k = 0$ for all $i \in \mathcal{A}(x_k)$

Conditions (A) and (B) $\Rightarrow p_k$ satisfies:

(A) $a_j^T p_k = -1$

(B) $A_k p_k = 0$
This is just a set of *linear equations* for the direction $p_k$.

Are these equations compatible?
\[
\begin{pmatrix}
A_k \\
a_j^T
\end{pmatrix}
\begin{pmatrix}
p_k \\
-1
\end{pmatrix} \leftarrow \text{vector of zeros}
\]

This is just a set of *linear equations* for the direction \( p_k \).

Are these equations compatible?

Yes! We will construct one solution (there may be others).
Write

\[ a_j = a_R + a_N \quad \text{with} \quad a_R \in \text{range}(A_k^T) \quad \text{and} \quad a_N \in \text{null}(A_k) \]

We know that
- \( a_R^T a_N = 0 \)
- \( a_N \neq 0 \)

(If \( a_N = 0 \), then

\[ a_j = a_R + a_N = a_R \in \text{range}(A_k^T) \]

which contradicts the assumption that \( a_j \) is independent of the rows of \( A_k \))
Define
\[ p_k = -\frac{1}{a_N^T a_N} a_N \neq 0 \]

Then \( p_k \in \text{null}(A_k) \), with
\[
a_j^T p_k = -\frac{1}{a_N^T a_N} a_j^T a_N = -\frac{1}{a_N^T a_N} (a_R + a_N)^T a_N
\]
\[
= -\frac{1}{a_N^T a_N} (a_R^T a_N + a_N^T a_N)
\]
\[
= -\frac{1}{a_N^T a_N} a_N^T a_N = -1
\]

All this implies that
\[
\begin{pmatrix} A_k \\ a_j^T \end{pmatrix} p_k = \begin{pmatrix} A_k p_k \\ a_j^T p_k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\]
**kth iteration: Step 3:**

Now we try to step along $p_k$ to the boundary of $a_j^T x \geq b_j$

We step to the hyperplane $a_j^T x = b_j$ as long we can stay feasible.
$k$th iteration: Step 3:

Find the step $\sigma_t$ to a constraint $a_t^T x \geq b_t$ that is blocking along $p_k$.

Since $a_t^T x \geq b_t$ is blocking, it must hold that

$$a_t^T (x_k + \sigma_t p_k) - b_t = 0 \implies a_t^T x \geq b_t \text{ is active at } x_k + \sigma_t p_k$$

Define $\alpha_k = \sigma_t$ and update the iterate and active set as

$$x_{k+1} = x_k + \alpha_k p_k$$

$$A_{k+1} = \begin{pmatrix} A_k \\ a_t \end{pmatrix} \text{ plus any other blocking constraints}$$

Finally, increment $k$ and continue at Step 1.
One loose end . . .

We have to show that

\[ \text{rank}(A_{k+1}) > \text{rank}(A_k) \]

**Result**

A blocking constraint \( a_t^T x \geq b_t \) is independent of the constraints in the active set \( A_k \).

**Proof:** Suppose that

\[ a_t = A_k^T w \quad \text{for some nonzero } w \]

Then

\[ p_k^T a_t = p_k^T A_k^T w = (A_k p_k)^T w \]
Repeating the last equation:

$$a_t^T p_k = p_k^T a_t = (A_k p_k)^T w$$

The vector $p_k$ was chosen so that $A_k p_k = 0$, giving

$$a_t^T p_k = (A_k p_k)^T w = 0$$

But a blocking constraint must satisfy $a_t^T p_k < 0$ because

$$\sigma_t = \sigma = \begin{cases} +\infty & \text{if } a_t^T p_k \geq 0 \\ \min_{i : a_i^T p_k < 0} \frac{r_i(\bar{x})}{(-a_i^T p_k)} & \text{otherwise} \end{cases}$$

$\Rightarrow$ a contradiction, and $a_t^T x \geq b_t$ must be independent of the active constraints.
It follows that

$$\text{rank}(A_{k+1}) > \text{rank}(A_k)$$

⇒ the algorithm above must terminate at a vertex in a finite number of steps.
Summary

We have proved that:

**Result (Existence of a vertex)**

If $F = \{ x : Ax \geq b \}$ has at least one point and $\text{rank}(A) = n$, then $F$ has at least one vertex.
\[ a_j^T x = b_j \]

\[ A_k x_k = b_k \]
\[ a_T^j x = b_j \]

\[ A_k x_k = b_k \]
\[ A_k x_k = b_k \]
\[ x_k + \sigma_j p_k \]
\[ a_j^T x = b_j \]
\[ a_j^T x = b_j \]

\[ a_t^T x = b_t \]

\[ A_k x_k = b_k \]
\[ a^T_T x = b_t \]

\[ a^T_j x = b_j \]

\[ x_{k+1} = x_k + \alpha_k p_k \]

\[ A_k x_k = b_k \]