Recap: LP formulations

Problems considered so far:

**ELP**
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x \in \mathbb{R}^n \quad Ax = b
\end{align*}
\]

**LP**
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\]

Now we consider a *mixture of constraint types*. 
Linear programs with mixed constraints

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \quad \text{equality constraints} \\
& \quad Dx \geq f, \quad \text{inequality constraints}
\end{align*}
\]

The dimensions are:

- \(A\) \(m \times n\) matrix
- \(b\) \(m\)-vector
- \(D\) \(m_D \times n\) matrix
- \(f\) \(m_D\)-vector
Optimality Conditions
Example: Consider inequalities and just one equality constraint:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a^T x = b, \quad D x \geq f
\end{align*}
\]

Write the equality constraint as two inequalities:

\[
a^T x \geq b \\
\text{and } a^T x \leq b, \quad \text{i.e., } (−a)^T x \geq −b
\]

If \(x^*\) is an optimal solution, then both

\[
a^T x \geq b \quad \text{and} \quad (−a)^T x \geq −b
\]

must be active at \(x^*\).
Suppose that \( x^* \) is a solution of the mixed-constraint problem:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a^T x \geq b, \quad (-a)^T x \geq -b, \quad Dx \geq f
\end{align*}
\]

This problem has all inequalities, so we can use existing theory.

Let \( D_a \) denote the matrix of active inequalities at \( x^* \).

The full active set for the mixed-constraint problem is

\[
\begin{pmatrix}
  a^T \\
  -(a^T) \\
  D_a
\end{pmatrix}
\begin{pmatrix}
  x^* \\
  b
\end{pmatrix}
= 
\begin{pmatrix}
  b \\
  -b \\
  f_a
\end{pmatrix}
\]
From the preceding slide:

\[ A_a'' = \begin{pmatrix} a^T \\ -a^T \\ D_a \end{pmatrix} \]

The optimality conditions \( A_a^T \lambda_a = c \), with \( \lambda_a \geq 0 \) are:

\[ c = (a \quad -a \quad D_a^T) \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ z_a^* \end{pmatrix}, \text{ with } \lambda_1^* \geq 0, \lambda_2^* \geq 0 \text{ and } z_a^* \geq 0 \]
From the preceding slide:

\[
c = \begin{pmatrix} a & -a & D_a^T \end{pmatrix} \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ z_a^* \end{pmatrix}, \text{ with } \lambda_1^* \geq 0, \lambda_2^* \geq 0 \text{ and } z_a^* \geq 0
\]

\[
c = a\lambda_1^* - a\lambda_2^* + D_a^T z_a^*
\]

\[
= a(\lambda_1^* - \lambda_2^*) + D_a^T z_a^*
\]

positive or negative

\[
= a\pi^* + D_a^T z_a^*, \text{ with } \pi^* = \lambda_1^* - \lambda_2^*
\]
From the preceding slide:

\[ c = a\pi^* + D_a^T z_a^*, \text{ with } z_a^* \geq 0 \]

\( \pi^* \) is the Lagrange multiplier for the constraint \( a^T x = b \).

\( \pi^* \) can be any sign.
Optimality conditions: active-set form

Result

Consider the linear program

\[
\text{minimize } \quad c^T x \\
\text{subject to } \quad Ax = b, \quad Dx \geq f
\]

Then \( x^* \) is a minimizer \textit{if and only if} \( Ax^* = b, \, Dx^* \geq f \), and there exist \( \pi^* \) and \( z_a^* \) such that

\[
c = A^T \pi^* + D_a^T z_a^*, \quad z_a^* \geq 0
\]

where \( D_a \) is the matrix of rows of \( D \) corresponding to inequality constraints active at \( x^* \).
### Optimality conditions: complementary slackness form

#### Result

Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \quad Dx \geq f
\end{align*}
\]

Then \(x^*\) is a minimizer if and only if \(Ax^* = b, \; Dx^* \geq f\), and there exist \(\pi^*\) and \(z^*\) such that

\[
\begin{align*}
c &= A^T \pi^* + D^T z^*, \quad z^* \geq 0 \\
z_i^*(d_i^T x^* - f_i) &= 0, \quad i = 1, \ldots, m_D
\end{align*}
\]
The Simplex Method
for Mixed Constraints
\[
\minimize_{x \in \mathbb{R}^n} \quad c^T x \\
\text{subject to } \quad Ax = b, \quad Dx \geq f
\]

Assume that \( A \) as full row rank.

The full constraint matrix is:

\[
\begin{pmatrix}
A \\
D
\end{pmatrix} \quad \text{← these rows are } \textit{always} \text{ active}
\]

\[
\begin{pmatrix}
A \\
D
\end{pmatrix} \quad \text{← a } \textit{subset} \text{ of these rows will be active}
\]
Definition of the working set:

\[ \mathcal{W}_k = \{ w_1, w_2, \ldots, w_{n-m} \} \]

\[ \text{rows of } D \]

\[ = \{ w_1, w_2, \ldots, w_{n-m} \} \]

The working-set defines a nonsingular matrix of the form

\[ A_k = \begin{pmatrix} A \\ D_k \end{pmatrix} \]
(Step 1) Computation of the multipliers:

\[ A_k^T \lambda_k = c \quad \text{with} \quad A_k = \begin{pmatrix} A \\ D_k \end{pmatrix} \]

Partition \( \lambda_k \) as

\[ \lambda_k = \begin{pmatrix} \pi_k \\ Z_k \end{pmatrix} \quad \leftarrow \text{multipliers for } A\mathbf{x} = b \]
\[ \leftarrow \text{multipliers for } D_k \mathbf{x}_k = f_k \]

This gives

\[ \left( A^T \quad D_k^T \right) \begin{pmatrix} \pi_k \\ Z_k \end{pmatrix} = c \]

\( \Rightarrow \) we need only check the signs of the multipliers for \( D_k \)

\( \Rightarrow \) choose \( s \) such that \([ Z_k ]_s < 0\)
(Step 2) Computation of the search direction:

\[
A_k p_k = \begin{pmatrix} 0 \\ e_s \end{pmatrix} = e_{m+s}
\]

Note that

\[
\begin{pmatrix} A \\ D_k \end{pmatrix} p_k = \begin{pmatrix} A p_k \\ D_k p_k \end{pmatrix} = \begin{pmatrix} 0 \\ e_s \end{pmatrix}
\]

\[\implies A p_k = 0 \quad \text{and} \quad D_k p_k = e_s\]
(Step 3) Computation of the maximum feasible step:

We need only check $Dx \geq f$ for the blocking constraint, i.e.,

$$\alpha_k = \min\{\sigma_i\}, \quad \text{with} \quad \sigma_i = \begin{cases} \frac{d_i^T x_k - f_i}{(-d_i^T p_k)} & \text{if } d_i^T p_k < 0 \\ +\infty & \text{otherwise} \end{cases}$$
(Step 4) Updates:

In Step 3, we will identify a blocking constraint $d_t^T x \geq f_t$.

\[ \mathcal{W}_k = \{w_1, w_2, \ldots, w_s, \ldots, w_{n-m}\} \]

\[ \uparrow \quad \text{moved off this constraint} \]

\[ \mathcal{W}_{k+1} = \{w_1, w_2, \ldots, t, \ldots, w_{n-m}\} \]

\[ \uparrow \quad \text{moved onto this constraint} \]

\[ A_{k+1} = \begin{pmatrix}
A \\
d_{w_1}^T \\
d_{w_2}^T \\
\vdots \\
d_t^T \\
\vdots \\
d_{w_{n-m}}^T
\end{pmatrix} \quad \leftarrow \text{row } m + s \]
Linear Programs in Standard Form
minimize \( c^T x \)

subject to \( Ax = b, \quad x \geq 0 \)

equality constraints \quad simple bounds

where \( A \) is \( m \times n \) with shape \( A = \) often, \( n \gg m \).

For standard form, we show that the two systems of order \( n \):

\[
A_T^k \lambda_k = c \quad \text{and} \quad A_k p_k = e_s
\]

are equivalent to two systems of order \( m \).
Every linear program can be written in standard form.

**Example:** Suppose the constraints are $Ax \geq b$.

$\Rightarrow$ there are no simple bounds (i.e., the $x_j$ are “free variables”.)
There are two steps involved in reformulating constraints in standard form:

- First, convert all the free variables into bounded variables
- Then convert the general inequalities into equalities
Suppose that we have $Ax \geq b$, with no bounds on $x$.

Define new nonnegative variables $u_i$ and $v_i$ such that

$$x_i = u_i - v_i, \quad u_i \geq 0, \quad v_i \geq 0$$

Then

$$Ax = A(u - v) = Au - Av = (A - A) \begin{pmatrix} u \\ v \end{pmatrix} \geq b$$

The objective becomes

$$c^T x = c^T (u - v) = c^T u - c^T v = (c^T - c^T) \begin{pmatrix} u \\ v \end{pmatrix}$$
This gives the linear program with $n' = 2n$ variables:

$$\begin{align*}
\text{minimize} & \quad c' Tx' \\
\text{subject to} & \quad A'x' \geq b', \quad x' \geq 0
\end{align*}$$

with

$$A' = (A - A), \quad x' = \begin{pmatrix} u \\ v \end{pmatrix}, \quad c' = \begin{pmatrix} c \\ -c \end{pmatrix} \quad \text{and} \quad b' = b$$

Now we assume that every variable is simply bounded, i.e., $x_i \geq 0$. 
Next we show how to convert inequalities to equalities.

Example:

\[
\begin{align*}
\text{minimize} & \quad -6x_1 - 9x_2 - 5x_3 \\
\text{subject to} & \quad 2x_1 + 3x_2 + x_3 \leq 5 \\
& \quad x_1 + 2x_2 + x_3 \geq 3 \\
& \quad x_1, \quad x_2, \quad x_3 \geq 0
\end{align*}
\]
Consider the constraint $2x_1 + 3x_2 + x_3 \leq 5$.

Consider a new variable $x_4$ and the equality constraint

$$2x_1 + 3x_2 + x_3 + x_4 = 5$$

For feasible $x_1$, $x_2$, $x_3$ and for $x_4 \geq 0$, it holds that

$$2x_1 + 3x_2 + x_3 \leq 5$$

The variable $x_4$ is called a *slack* variable.
Now consider the constraint $x_1 + 2x_2 + x_3 \geq 3$.

Consider a new variable $x_5$ and the equality constraint

$$x_1 + 2x_2 + x_3 - x_5 = 3$$

For feasible $x_1, x_2, x_3$ and $x_5 \geq 0$, then

$$x_1 + 2x_2 + x_3 \geq 3$$

$x_5$ is called a *surplus variable*. 
The final problem in standard form is:

\[
\begin{align*}
\text{minimize} & \quad -6x_1 - 9x_2 - 5x_3 \\
\text{subject to} & \quad 2x_1 + 3x_2 + x_3 + x_4 = 5 \\
& \quad x_1 + 2x_2 + x_3 - x_5 = 3 \\
& \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \geq 0
\end{align*}
\]

- “Slack variable” is a generic name for both slack and surplus variables.
- An equality constraint \( a^T x = b \) needs neither a slack nor a surplus variable.
Optimality Conditions
for Standard Form
minimize \( c^T x \) \( \quad \text{subject to} \quad \begin{align*} x \in \mathbb{R}^n \quad & \quad \text{minimize} \quad c^T x \quad \text{subject to} \quad \begin{align*} x \in \mathbb{R}^n \quad & \quad \text{subject to} \quad A x = b \quad \Rightarrow \quad A x = b \quad \Rightarrow \quad Dx = f \quad \text{with} \quad \begin{cases} D = I \\ f = 0 \end{cases} \end{align*} \end{align*} \}

The “full” vector of residuals at a feasible point is:

\[
\begin{pmatrix} A \\ D \end{pmatrix} x - \begin{pmatrix} b \\ f \end{pmatrix} = \begin{pmatrix} A \\ l \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} Ax - b \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}
\]

\[\Rightarrow\] the inequality-constraint residuals are just the values of \( x \).
Consider the standard-form linear program

\[
\minimize_{x \in \mathbb{R}^n} \quad c^T x
\]
subject to \[ Ax = b, \quad x \geq 0 \]

Then \( x^* \) is a minimizer if and only if

(A) \( Ax^* = b, \ x^* \geq 0; \)
(B) \( c = A^T \pi^* + z^*, \quad z^* \geq 0; \)
(C) \( z^*_i x^*_i = 0 \) for \( i = 1, \ldots, n. \)

The vector \( z^* \triangleq c - A^T \pi^* \) is called the vector of reduced costs.

The reduced costs are the multipliers associated with \( x \geq 0. \)