General Form of a Linear Program

A general linear program (LP) can have thousands of variables and constraints. We will assume problems in the following form.

Assume that there are $n$ variables,

$$x_1, x_2, \ldots, x_n$$

and $m$ constraints:

- **minimize**
  
  \[
  c_1 x_1 + c_2 x_2 + \cdots + c_n x_n
  \]

- **subject to**
  
  \[
  a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \geq b_1 \\
  a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \geq b_2 \\
  \vdots \\
  a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \geq b_m
  \]
Remember your linear algebra?

\[ c_1x_1 + c_2x_2 + \cdots + c_nx_n = \sum_{i=1}^{n} c_ix_i = c^Tx \]

where \( x \) and \( c \in \mathbb{R}^n \)

Similarly, the \( i \)th constraint is

\[ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i \implies a_i^Tx \geq b_i \]

where

\[
a_i = \begin{pmatrix}
a_{i1} \\
a_{i2} \\
\vdots \\
a_{in}
\end{pmatrix}
\]

The generic LP can be written as

\[
\begin{align*}
\text{minimize} & \quad c^Tx \\
\text{subject to} & \quad a_1^Tx \geq b_1 \\
& \quad a_2^Tx \geq b_2 \\
& \quad \vdots \\
& \quad a_m^Tx \geq b_m
\end{align*}
\]
Define the \( m \times n \) matrix \( A \) and \( m \)-vector \( b \) such that

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} = \begin{pmatrix}
a_1^T \\
a_2^T \\
\vdots \\
a_m^T
\end{pmatrix}
\]

and

\[
b = \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix}
\]

All-inequality linear program

**Linear program in generic form**

\[
\begin{aligned}
\text{(LP)} \quad & \text{minimize} & & c^T x \\
\text{subject to} & & & Ax \geq b
\end{aligned}
\]

This is a “generic” linear program in so-called \textit{all-inequality form}.

Note, minimizing \( c^T x \) is equivalent to maximizing \(-c^T x\).
All-inequality linear program

How would we convert the JuiceCo problem?

\[
\begin{align*}
\text{maximize} & \quad 3x + 4y \\
\text{subject to} & \quad x + 2y \leq 100 \\
 & \quad 3x + 2y \leq 200 \\
 & \quad x \geq 0 \\
 & \quad y \geq 0
\end{align*}
\]
Properties of Linear Constraints

Consider a single linear inequality constraint \( a^T x \geq b \).

**Definition**

Any point \( x_0 \) such that \( a^T x_0 \geq b \) is said to be a *feasible point* for the inequality \( a^T x \geq b \).

The feasible points of an inequality constraint form the set

\[
C = \{ x \in \mathbb{R}^n : a^T x \geq b \}
\]

**Definition**

A constraint \( a^T x \geq b \) is *satisfied* or *feasible* at \( x_0 \) if \( a^T x_0 \geq b \), or, equivalently, if \( x_0 \in C \), with \( C = \{ x \in \mathbb{R}^n : a^T x \geq b \} \).

**Definition**

A constraint \( a^T x \geq b \) is *strictly satisfied* or *strictly feasible* at \( x_0 \) if \( a^T x_0 > b \).
Definition
A constraint $a^T x \geq b$ is **violated** or **infeasible** at $x_0$ if $a^T x_0 < b$.

Definition
A constraint $a^T x \geq b$ is **active** at $x_0$ if $a^T x_0 = b$. ($a^T x \geq b$ is sometimes called a **binding constraint** at $x_0$.)

Definition
Two constraints are **equivalent** if they have the same set of feasible points.

Examples:
- $a^T x \geq b$ is equivalent to $(\gamma a)^T x \geq \gamma b$ for $\gamma > 0$
- $a^T x \leq b$ is equivalent to $(\gamma a)^T x \geq \gamma b$ for $\gamma < 0$

In particular,
$$d^T x \leq \delta$$ is equivalent to $$-d^T x \geq -\delta$$
Geometry Review

Definition

Consider the set $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = b\}$.

- If $n = 2$ then $\mathcal{H}$ is called a line.
- If $n = 3$ then $\mathcal{H}$ is called a plane.
- If $n > 3$ then $\mathcal{H}$ is called a hyperplane.

For simplicity, we refer to $\mathcal{H}$ as a hyperplane even if $n = 2$ or $n = 3$.

A constraint $a^T x \geq b$ splits $\mathbb{R}^n$ into two half-spaces, one containing feasible points, the other containing infeasible points.

$\mathcal{H}$ consists of the boundary points of the feasible half-space.
Definition
A point \( y \) is said to lie on a hyperplane \( \mathcal{H} = \{ x : a^T x = b \} \) if \( a^T y = b \).

Definition
A vector \( p \) joining two points \( v \) and \( w \) lying on a hyperplane is said to lie in a hyperplane.

As \( w \) and \( y \) lie on the hyperplane \( a^T x = b \), it must hold that

\[
\begin{align*}
  a^T y &= b \quad \text{and} \quad a^T w = b \\
  \end{align*}
\]

We have \( p \) joining \( y \) and \( w \), i.e., \( p = y - w \), so that

\[
\begin{align*}
  a^T p &= a^T (y - w) \\
  &= a^T y - a^T w \\
  &= b - b \\
  &= 0 \\
\end{align*}
\]

\( \Rightarrow \) \( a \) is perpendicular to every \( p \) that lies in the hyperplane

\( \Rightarrow \) \( a \) is the normal vector for the hyperplane.
Hyperplanes with different right-hand sides are parallel (same normal vector).
The nonnegativity constraint $x_i \geq 0$ has a very simple form of hyperplane:

$$\mathcal{H} = \{x : x_i = 0\} = \{x : e_i^T x = 0\}$$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \leftarrow \text{row } i$$

Some linear algebra review

Let $x$ and $y$ be $n$-vectors.

- $\sum_{i=1}^n x_i y_i = x^T y = y^T x$ is the inner product of $x$ and $y$

- $\left( \sum_{i=1}^n x_i^2 \right)^{1/2} = (x^T x)^{1/2}$ is the Euclidean length of $x$

The Euclidean length of $x$ is called the norm of $x$ and is denoted by $\|x\|$.

- If $\alpha$ is a scalar, then $\|\alpha x\| = |\alpha| \|x\|$

- If $x^T y = 0$, then $x$ and $y$ are orthogonal
Distance of a point to a hyperplane

The normal vector \( a \) is orthogonal to the hyperplane \( a^T x = b \). The closest point to an arbitrary point \( x_0 \) is \( x_0 + \alpha a \) for some appropriate value of \( \alpha \).

The distance from \( x_0 \) to \( x_0 + \alpha a \) is

\[
\|d\| = \|x_0 + \alpha a - x_0\| = \|\alpha a\| = |\alpha| \|a\|
\]

What’s \( \alpha \)?

Distance of a point to a hyperplane

\( x_0 + \alpha a \) lies on the hyperplane \( a^T x = b \).

\[
b = a^T (x_0 + \alpha a) = a^T x_0 + \alpha a^T a
\]

\( \Rightarrow \) \( \alpha = \frac{a^T x_0 - b}{\|a\|^2} \)

Therefore, the distance of \( x_0 \) to the hyperplane is

\[
\|d\| = |\alpha| \|a\| = \left| \frac{a^T x_0 - b}{\|a\|} \right|
\]
Distance of a point to a hyperplane

Result

Given $x_0 \in \mathbb{R}^n$ and a hyperplane $a^T x = b$, the quantity

$$\frac{|a^T x_0 - b|}{\|a\|}$$

measures the perpendicular distance of $x_0$ to $a^T x = b$. 