Math 171A: Linear Programming

Lecture 21
Simplex Method for Standard-Form LPs

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Recap

Result

$x^*$ is a minimizer for the standard-form LP

$$
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \quad x \geq 0
\end{align*}
$$

if and only if

(A) $Ax^* = b, \ x^* \geq 0$;
(B) $c = A^T \pi^* + z^*, \ z^* \geq 0$;
(C) $z_i^* x_i^* = 0$ for $i = 1, \ldots, n$.

The vector $z^* \triangleq c - A^T \pi^*$ is called the vector of reduced costs. The reduced costs are the multipliers associated with $x \geq 0$. $\pi^*$ are the multipliers associated with $Ax = b$. 
We define a basic set $B$ such that

$$AP = \begin{pmatrix} B \\ N \end{pmatrix}$$

where $B$ is nonsingular and $m \times m$, $N$ contains the remaining columns of $A$ forming the nonbasic set $\mathcal{N}$, and $P$ is a column permutation matrix.

A vertex of the feasible region: $F = \{ x : Ax = b, \ x \geq 0 \}$ is a nonnegative basic solution of $Ax = b$.

To find a vertex, we choose a basic set $B$ such that $B$ is nonsingular, and solve $Bx_B = b$. If $x_B \geq 0$, then we have a vertex.
The permutation matrix $P$ moves the basic columns to the front.

**Result**

$P$ is an **orthogonal matrix**, i.e., $P^TP = I = PP^T$. 
Example: We define the basic set $B = \{2, 5\}$, the nonbasic set is $N = \{1, 3, 4\}$.

$$AP = \begin{pmatrix} 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & -1 \end{pmatrix} \begin{bmatrix} \#2 & \#5 & \#1 & \#3 & \#4 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 2 & 1 & 1 \\ 2 & -1 & 1 & 1 & 0 \end{pmatrix} = (B \ N)$$

This gives

$$B = \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \text{ and } N = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
The column permutation applied to a row vector produces a row vector with “basic part” first.

\[ \begin{align*}
  c^T P &= \left( c_B^T \quad c_N^T \right) \implies P^T c = \begin{pmatrix} c_B \\ c_N \end{pmatrix} \\
  x^T P &= \left( x_B^T \quad x_N^T \right) \implies P^T x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} \\
  p^T P &= \left( p_B^T \quad p_N^T \right) \implies P^T p = \begin{pmatrix} p_B \\ p_N \end{pmatrix}
\end{align*} \]

The same goes for matrices:

\[ \begin{align*}
  AP &= \begin{pmatrix} B & N \end{pmatrix} \\
  I^k P &= \begin{pmatrix} 0 & I_{n-m} \end{pmatrix} \ vraught \begin{pmatrix} 0 & I_n \end{pmatrix}
\end{align*} \]

where \( I^k \) is formed from \( n - m \) rows \( \{ e_j^T \} \) of \( I_n \) with \( j \in \mathcal{N} \).
Simplex Method for Standard Form
The nonsingular matrix associated with the mixed-constraint simplex method is:

\[ A_k = \begin{pmatrix} A \\ I^k \end{pmatrix} \leftarrow n - m \text{ rows of } I_n \text{ (nonbasics)} \]

and \[ b_k = \begin{pmatrix} b \\ 0 \end{pmatrix} \leftarrow n - m \text{ components from } 0_n \]
The permutation $P$ moves the basic variables to the front, so that

$\begin{pmatrix} A \\ I_k \end{pmatrix} P = A_k P = \begin{pmatrix} B & N \\ 0 & I_{n-m} \end{pmatrix}$

Result

The matrix $A_k$ is nonsingular \textit{if and only if} $B$ is nonsingular.

Proof: Homework.
The vertex \( x_k \) defined by the simplex working set satisfies

\[
A_k x_k = b_k
\]

\[
\implies A_k P P^T x_k = b_k
\]

\[
\implies (A_k P)(P^T x_k) = b_k
\]

\[
\implies \begin{pmatrix} B & N \\ 0 & I_N \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}
\]

\[
B x_B + N x_N = b
\]

\[
x_N = 0 \implies B x_B = b \quad \text{and} \quad x_B \geq 0
\]
The simplex multipliers satisfy $A_k^T \lambda_k = c$.

Multiplying by $P^T$ gives

\[ P^T A_k^T \lambda_k = P^T c \]

\[ \implies (A_k P)^T \lambda_k = P^T c \]

\[ \implies \begin{pmatrix} B^T & 0 \\ N^T & I_N \end{pmatrix} \lambda_k = \begin{pmatrix} c_B \\ c_N \end{pmatrix} \]

If the multipliers are partitioned as

\[ \lambda_k = \begin{pmatrix} \pi \\ Z_N \end{pmatrix} \quad \leftarrow \text{multipliers for } Ax = b \]

\[ \leftarrow \text{multipliers for } x \geq 0 \]

then

\[ \begin{pmatrix} B^T & 0 \\ N^T & I_N \end{pmatrix} \begin{pmatrix} \pi \\ Z_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix} \]
From the preceding slide:

\[
\begin{pmatrix}
B^T & 0 \\
N^T & I_N \\
\end{pmatrix}
\begin{pmatrix}
\pi \\
z_N \\
\end{pmatrix}
= 
\begin{pmatrix}
c_B \\
c_N \\
\end{pmatrix}
\]

This implies that \( \pi \) and \( z_N \) satisfy

\[
B^T \pi = c_B \\
z_N = c_N - N^T \pi
\]

Recall that \( z = c - A^T \pi \) are the \textit{reduced costs}.

\[\Rightarrow \]
\( z_N \) are the reduced costs for the \textit{nonbasic variables}. 
Observe that

\[ P^T z = \left( \begin{array}{c} z_B \\ z_N \end{array} \right) = P^T (c - A^T \pi) \]

\[ = P^T c - P^T A^T \pi \]

\[ = P^T c - (AP)^T \pi \]

\[ = \begin{pmatrix} c_B \\ c_N \end{pmatrix} - \begin{pmatrix} B^T \\ N^T \end{pmatrix} \pi \]

\[ = \begin{pmatrix} c_B - B^T \pi \\ c_N - N^T \pi \end{pmatrix} \]

but \( B^T \pi = c_B \Rightarrow c_B - B^T \pi = 0. \)

\[ \Rightarrow \] the reduced costs associated with the basic variables are zero.
Simplex for Standard-Form LP: $k$th iteration

**Step 1:** Check for optimality (Implicitly solve $A_k^T \lambda_k = c$)

Solve $B^T \pi = c_B$ and set $z_N = c_N - N^T \pi$.

If $[z_N]_i \geq 0$ for $i = 1, 2, \ldots, n - m$, then STOP.

Otherwise, define $[z_N]_s = \min(z_N)$.

The $s$-th nonbasic variable (i.e., $x_{\nu_1}$) will become basic.

$$\mathcal{N} = \{\nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_{n-m}\}$$

$s$th element of $\mathcal{N}$
Step 2: Compute the search direction (Implicitly solve $A_k p_k = e_{m+s}$)

\[
\begin{pmatrix} B & N \\ 0 & I_N \end{pmatrix} \begin{pmatrix} p_B \\ p_N \end{pmatrix} = \begin{pmatrix} 0 \\ e_s \end{pmatrix}
\]

A step along $p_k$ increases $x_{\nu_s}$ (i.e., moves off $x_{\nu_s} = 0$) but keeps other nonbasics fixed at 0.
Step 2: (continued) Compute the search direction

Solve

\[ Bp_B + Np_N = 0 \]
\[ p_N = e_s \]

\[ Bp_B = -Np_N = -Ne_s = -(\text{sth column of } N) \]
\[ = -(\text{column } \nu_s \text{ of } A) \]
\[ = -a_{\nu_s} \]

⇒ we solve \( Bp_B = -a_{\nu_s} \)
Step 3: Step to an adjacent vertex

If we take a step $\alpha$ along the vector

$$P^T p = \begin{pmatrix} p_B \\ p_N \end{pmatrix} = \begin{pmatrix} p_B \\ e_s \end{pmatrix}$$

Then the basic variables change to $x_B + \alpha p_B$

$\Rightarrow$ all the basic variables change
Step 3: (continued) Step to an adjacent vertex

What happens to the nonbasic variables?

$$x_N + \alpha p_N = x_N + \alpha e_s = x_N + \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{row } s$$

$$\Rightarrow$$ all nonbasics remain at 0 except $x_{\nu_s}$, which wants to \textit{increase}.
Step 3: (continued) Step to an adjacent vertex

The changed variables are $x_B$ and $x_{\nu_s}$ (which increases from 0 to $\alpha$)

We must ensure that

$$x_B + \alpha p_B \geq 0$$

$$\sigma_i = \begin{cases} 
\frac{[x_B]_i}{-[p_B]_i} & \text{if } [p_B]_i < 0 \\
+\infty & \text{if } [p_B]_i \geq 0 
\end{cases}$$

This is known as the min ratio test.

Define $\alpha = \min\{\sigma_i\}$

If $\alpha = +\infty$, the LP is unbounded, STOP.
Step 4: Update

If $\alpha = \sigma_t$ then

$$[x_B]_t + \sigma_t[p_B]_t = 0 \implies \text{the } t\text{-th basic becomes nonbasic}$$

$t$ points to index $\beta_t$ of $B$

$s$ points to index $\nu_s$ of $N$

$$[x_N]_s \rightarrow \alpha \implies x_{\nu_s} \text{ goes from nonbasic to basic}$$

$$[x_B]_t \rightarrow 0 \implies x_{\beta_t} \text{ goes from basic to nonbasic}$$
Step 4: (continued) Update

\[ B \leftarrow \{ \beta_1, \beta_2, \ldots, \beta_{s-1}, \nu_s, \beta_{s+1}, \ldots, \beta_m \} \]
\[ \uparrow \]
\[ \text{moved from nonbasic set} \]

\[ N \leftarrow \{ \nu_1, \nu_2, \ldots, \nu_{t-1}, \beta_t, \nu_{t+1}, \ldots, \nu_{n-m} \} \]
\[ \uparrow \]
\[ \text{moved from basic set} \]

\[ \Rightarrow B \text{ and } N \text{ exchange indices} \]

\[ \Rightarrow \text{column } t \text{ of } B \text{ is replaced by column } \nu_s \text{ of } A \]

\[ \Rightarrow \text{Solve } Bx_B = b \text{ with the new } B \]
$B$ and $N$ exchange just one index

$\Rightarrow$ it is unnecessary to re-solve $Bx_B = b$

Only one column of $B$ changes $\Rightarrow$ only one element of $x_B$ changes

$\Rightarrow$ we can replace $[x_B]_t (= 0)$ with $[x_N]_s (= \alpha)$

$\Rightarrow$ $x_B \leftarrow x_B + \alpha p_B$, $[x_B]_t \leftarrow \alpha$
Summary

- Two $m \times m$ systems are solved:
  \[ B^T \pi = c_B \]
  \[ Bp_B = -a_{\nu s} \]

- $z_B = 0$, $x_N = 0$, $p_N = e_s$

- Only $\pi$, $z_N$, $x_B$ and $p_B$ need be stored
Summary: Simplex method for standard form

Solve $B^T \pi = c_B$;
Compute $z_N = c_N - N^T \pi$;
$[z_N]_s = \min(z_N)$;
if $[z_N]_s \geq 0$ then stop;
Solve $Bp_B = -a_{\nu_s}$;

\[
\sigma_i = \begin{cases} 
\frac{[x_B]_i}{-[p_B]_i} & \text{if } [p_B]_i < 0; \\
-\frac{[p_B]_i}{+[p_B]_i} & \text{if } [p_B]_i \geq 0; \\
+\infty & \text{if } [p_B]_i < 0; \\
\end{cases}
\]

$\sigma_t = \min\{\sigma_i\}; \quad \alpha = \sigma_t$;
if $\alpha = +\infty$ then stop;
Exchange index $\beta_t$ of $B$ with index $\nu_s$ of $N$;
$x_B \leftarrow x_B + \alpha p_B$; $[x_B]_t \leftarrow \alpha$;