Recap: Duality

Problem conversion involves defining a primal linear program and converting it into another dual linear program.

Primal in all-inequality form:

\[
(P): \begin{array}{l}
\text{minimize} \quad c^T x \\
\text{subject to} \quad Ax \geq b
\end{array}
\]

\[
(D): \begin{array}{l}
\text{maximize} \quad b^T \lambda \\
\text{subject to} \quad A^T \lambda = c, \quad \lambda \geq 0
\end{array}
\]

The appropriate simplex method applied to (P) and (D) both results in systems of size \(m \times m\).

(The constraints in (D) are \(n \times m - A^T \lambda = c\))

Weak duality:

If \(x^*\) is a feasible bounded solution of the primal and \(\lambda^*\) is a solution of the dual, then \(c^T x^* \geq b^T \lambda^*\).

Strong duality:

If \(x^*\) is a feasible bounded solution of the primal, then there exists a feasible bounded solution \(\lambda^*\) of the dual with \(b^T \lambda^* = c^T x^*\).

Summary

\[
(P): \begin{array}{l}
\text{minimize} \quad c^T x \\
\text{subject to} \quad Ax \geq b
\end{array}
\]

\[
(D): \begin{array}{l}
\text{maximize} \quad b^T \lambda \\
\text{subject to} \quad A^T \lambda = c, \quad \lambda \geq 0
\end{array}
\]

Solution \(x^*\) \quad Solution \(\lambda^*\)
Multipliers \(\lambda^*\) \quad \pi\text{-values} \(\pi^*\)
Residual \(r^*\) \quad Reduced costs \(z^*\)
Working set \(A_\ell\) \quad Basis \(B\)

The primal and dual solutions satisfy the following:

\[x^* = -\pi^*\]

primal multipliers \(\lambda^* = \text{dual solution} \quad \lambda^*\) \((\lambda^*_\ell = \lambda^*_B)\)
\[r^* = z^*\]
\[A_\ell = B^T\]
How is the basic set from the dual standard-form LP related to the primal all-inequality LP?

Strong duality implies that

\[
\text{Final dual basic set} = \text{Final primal working set}
\]

\[
A^T \downarrow B \triangleq A^T \ell \iff A \rightarrow \ell \Rightarrow A_\ell = B^T
\]

defines an optimal working-set matrix for the primal.

The strong duality implies that if one of the primal or dual problems has a \textit{bounded} optimal value, then so does the other (because \(c^T x^* = b^T \lambda^*\)).

What happens if the primal or dual problem is \textit{unbounded}?

**Unbounded Primal Problem**

The primal problem is:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\]

Suppose that the primal constraints are feasible, but \(c^T x\) is \textit{unbounded} in the feasible region.

If \(c^T x\) is unbounded, there must exist a direction \(p\) along which \(c^T x\) is \textit{decreasing} and every constraint is increasing, i.e.,

\[
c^T p < 0 \quad \text{and} \quad Ap \geq 0
\]
From the preceding slide, $c^T x$ is unbounded if there is a $p$ such that $c^T p < 0$ and $A p \geq 0$.

Farkas' lemma states that $c^T p < 0$ and $A p \geq 0$ if and only if there is no $\lambda$ such that $c = A^T \lambda$ and $\lambda \geq 0$.

$\Rightarrow$ there is no feasible point for the dual constraints.

**Unbounded Dual Problem**

The dual problem is:

$$\max_{\lambda \in \mathbb{R}^n} b^T \lambda$$

subject to $A^T \lambda = c, \; \lambda \geq 0$

Now suppose that the dual constraints are feasible, but $b^T \lambda$ is unbounded in the dual feasible region.

If $b^T \lambda$ is unbounded, there must exist a direction $q$ along which $b^T \lambda$ is increasing and every constraint is increasing, i.e.,

$$b^T q > 0 \quad \text{and} \quad A^T q = 0, \; q \geq 0$$

By writing $A^T q = 0$ as two inequalities $A^T q \geq 0$ and $A^T q \leq 0$, we can write

$$b^T q > 0 \quad \text{and} \quad A^T q = 0, \; q \geq 0$$

as

$$-b^T q < 0 \quad \text{and} \quad \begin{pmatrix} A^T \\ -A^T \\ l_m \end{pmatrix} q \geq 0$$

Now apply Farkas Lemma with

"$c'' = -b$, "$A'' = \begin{pmatrix} A^T \\ -A^T \\ l_m \end{pmatrix}$ and "$p'' = q"$

Farkas Lemma states that:

$$-b^T q < 0 \quad \text{and} \quad \begin{pmatrix} A^T \\ -A^T \\ l_m \end{pmatrix} q \geq 0$$

if and only if there are no vectors $x_-, x_+$ and $s$ such that

$$\begin{pmatrix} A & -A & l_m \end{pmatrix} \begin{pmatrix} x_- \\ x_+ \\ s \end{pmatrix} = -b, \; \text{and} \; \begin{pmatrix} x_- \\ x_+ \\ s \end{pmatrix} \geq 0$$
\[ Ax_+ - Ax_- + s = -b \quad \text{and} \quad \begin{pmatrix} x_- \\ x_+ \\ s \end{pmatrix} \geq 0. \]

If \( x \triangleq x_+ - x_- \), then there is no \( x \) such that \( Ax - s = b \) with \( s \geq 0 \).

\[ \Rightarrow \text{there is no } x \text{ such that } Ax \geq b. \]

\textbf{Result}

Consider the primal linear program (\( P \)) and its corresponding dual (\( D \)):

\[
(P): \begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \geq b
\end{array} \\
(D): \begin{array}{ll}
\text{maximize} & b^T \lambda \\
\text{subject to} & A^T \lambda = c, \quad \lambda \geq 0
\end{array}
\]

Let \( U_P \) and \( U_D \) denote the sets of unbounded primal and dual directions

\[ U_P = \{ p \in \mathbb{R}^n : c^T p < 0, \quad Ap \geq 0 \} \quad \text{and} \quad U_D = \{ q \in \mathbb{R}^m : b^T q > 0, \quad A^T q = 0, \quad q \geq 0 \}. \]

\textbf{(A)} The dual is feasible if and only if \( U_P \) is empty.

The primal is feasible if and only if \( U_D \) is empty.

\textbf{(B)} If the primal is feasible, then \( c^T x^* > -\infty \) if and only if \( U_P \) is empty.

If the dual is feasible, then \( b^T \lambda^* < \infty \) if and only if \( U_D \) is empty.

\textbf{(C)} If the primal is feasible and unbounded, then the dual is infeasible.

If the dual is feasible and unbounded, then the primal is infeasible.

\textbf{(D)} If either the primal (or dual) problem has a bounded feasible solution then so does the corresponding dual (or primal), in which case \( c^T x^* = b^T \lambda^* \).

The possible combinations of primal and dual properties are summarized below.

\begin{center}
\begin{tabular}{ccc}
\textbf{Primal} & \textbf{Dual} \\
Feasible; bounded objective & \iff & Feasible; bounded objective \\
Feasible; unbounded objective & \implies & Infeasible \\
Infeasible & \iff & Feasible; unbounded objective \\
Infeasible & ? & Infeasible
\end{tabular}
\end{center}

The only case not treated so far is the last, which says that both primal \textit{and} dual problems may be infeasible.

\textbf{Primal and Dual Infeasibility Example}

Consider, the \textit{infeasible} primal problem

\[
\begin{aligned}
\text{minimize} & \quad x_1 + 2x_2 + 3x_3 \\
\text{subject to} & \quad x_1 + x_2 + x_3 \geq 1 \\
& \quad -x_1 - x_2 - x_3 \geq 1
\end{aligned}
\]

for which

\[
c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \end{pmatrix}
\]

The dual constraints are \( A^T \lambda = c \), \( \lambda \geq 0 \), i.e.,

\[
\lambda_1 - \lambda_2 = 1, \quad \lambda_1 - \lambda_2 = 2, \quad \lambda_1 - \lambda_2 = 3
\]

which are inconsistent \( \Rightarrow \) the dual is \textit{also infeasible}. 

\textbf{Unbounded Dual Problem} \implies \textbf{Infeasible Primal Problem}
Now we start with the primal problem in standard form

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \quad x \geq 0
\end{align*}
\]

where \( A \) has \( m \) rows.

The optimality conditions are:

\[
\begin{align*}
Ax^* &= b, \quad x^* \geq 0 \\
c &= A^T \pi^* + z^*, \quad z^* \geq 0 \\
z^* \cdot x^* &= 0
\end{align*}
\]

where

\[
[z^* \cdot x^*]_i = z^*_i x^*_i
\]

Consider the conditions

\[
A^T \pi^* + z^* = c, \quad z^* \geq 0
\]

\( \Rightarrow \) \( \pi^* \) satisfies the inequality constraints \( A^T \pi \leq c \)

This suggests the dual problem:

\[
\begin{align*}
\text{maximize} & \quad b^T \pi \\
\text{subject to} & \quad A^T \pi \leq c
\end{align*}
\]

Apply the all-inequality-form simplex method to the dual

\[
\begin{align*}
\text{maximize} & \quad b^T \pi & \Rightarrow \quad \text{minimize} & \quad -b^T \pi \\
\text{subject to} & \quad A^T \pi \leq c & \text{subject to} & \quad (-A)^T \pi \geq -c
\end{align*}
\]

The working-set for the dual defines a nonsingular subset of the rows of \(( -A)^T \).

\( \Rightarrow \) the working-set for the dual defines a basis for the primal.
The dual problem from the previous slide is:

\[
\begin{align*}
\text{minimize} & \quad -b^T \pi \\
\text{subject to} & \quad (-A)^T \pi \geq -c
\end{align*}
\]

The optimality conditions for the dual all-inequality problem are

(A) \( -A^T \pi^* \geq -c \),
(B) \( -b = -A \lambda^*, \lambda^* \geq 0 \),
(C) \( \lambda^* \cdot (-A^T \pi^* + c) = 0 \)

where \( \lambda^* \) are the multipliers for the inequalities \(-A^T \pi \geq -c\).

If \( x^* = \lambda^* \), then

(B) \( \implies Ax^* = b \), \( x^* \geq 0 \)

If we let \( z^* \triangleq c - A^T \pi^* \), then

(C) \( \implies x^* \cdot (A^T \pi^* - c) = x^* \cdot z^* = 0 \)

and

(A) \( \implies z^* = c - A^T \pi^* \geq 0 \).

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**Summary**

(P): minimize \( c^T x \)

subject to \( Ax = b, \ x \geq 0 \)

Solution \( x^* \)

Multipliers \( \pi^* \)

Reduced costs \( z^* \)

Basis \( B \)

The primal and dual solutions satisfy the following:

\( x^* = \lambda^* \)

optimal primal multipliers \( \pi^* = \) dual optimal solution \( \pi^* \)

\( z^* = r^* \)

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**Result**

Consider the primal linear program (P) and its corresponding dual (D):

(P): minimize \( c^T x \)

subject to \( Ax = b, \ x \geq 0 \)

Solution \( x^* \)

Multipliers \( \pi^* \)

Reduced costs \( z^* \)

Basis \( B \)

(D): maximize \( b^T \pi \)

subject to \( A^T \pi \leq c \)

Solution \( \pi^* \)

Multipliers \( \lambda^* \)

Residual \( r^* \)

Working set \( (-B)^T \)

Let \( U_P \) and \( U_D \) denote the sets of unbounded primal and dual directions

\( U_P = \{ p \in \mathbb{R}^n : c^T p < 0, \ Ap = 0, \ p \geq 0 \} \)

and \( U_D = \{ q \in \mathbb{R}^m : b^T q > 0, \ A^T q \leq 0 \} \)

(A) The dual is feasible if and only if \( U_P \) is empty.

The primal is feasible if and only if \( U_D \) is empty.

(B) If the primal is feasible, then \( c^T x^* > -\infty \) if and only if \( U_P \) is empty.

If the dual is feasible, then \( b^T \pi^* < \infty \) if and only if \( U_D \) is empty.

(C) If the primal is feasible and unbounded, then the dual is infeasible.

If the dual is feasible and unbounded, then the primal is infeasible.

(D) If either the primal (or dual) problem has a bounded feasible solution then so does the corresponding dual (or primal), in which case \( c^T x^* = b^T \pi^* \).
Summary: Duality

Primal in all-inequality form:

\[(P): \begin{array}{l}
\text{minimize } c^T x \\
\text{subject to } Ax \geq b
\end{array} \quad \begin{array}{l}
\text{(D): } \\
\text{maximize } b^T \lambda \\
\text{subject to } A^T \lambda = c, \lambda \geq 0
\end{array}\]

Primal in standard form:

\[(P): \begin{array}{l}
\text{minimize } c^T x \\
\text{subject to } Ax = b, \ x \geq 0
\end{array} \quad \begin{array}{l}
\text{(D): } \\
\text{maximize } b^T \pi \\
\text{subject to } A^T \pi \leq c
\end{array}\]

Example:

The primal in standard form:

\[\begin{array}{l}
\text{minimize } 2x_1 + x_2 \\
\text{subject to } x_1 + x_2 - x_3 = 1 \\
-2x_1 - x_2 = -2 \\
-x_1 + 2x_2 - x_3 = -2 \\
x_i \geq 0, \ i = 1, \ldots, 5
\end{array}\]

The optimal solution is \(x^* = (0 \ 1 \ 2 \ 3)^T\) with \(\pi^* = (1 \ 0 \ 0 \ 0)^T\) and a final basic set of \(\mathcal{B}^* = \{2, 4, 5\}\).

The dual in all-inequality form:

\[\begin{array}{l}
\text{minimize } -\lambda_2 + 2\lambda_3 \\
\text{subject to } -\lambda_2 + \lambda_3 \geq -2 \\
-\lambda_2 - \lambda_3 \geq -1 \\
\lambda_1 \geq 0 \\
\lambda_2 \geq 0 \\
\lambda_3 \geq 0
\end{array}\]

The optimal solution is \(\lambda^* = (1 \ 0 \ 0 \ 0)^T\) with multipliers \((1 \ 2 \ 3)^T\) and a final working set of \(\mathcal{W}^* = \{2, 4, 5\}\).

We can "scatter" multipliers to get \(\lambda^* = (1 \ 0 \ 0 \ 1 \ 0)^T\).

The constraint matrix is \(5 \times 2\). All-inequality simplex will solve 2 \times 2 systems.

Example:

The primal in all-inequality form:

\[\begin{array}{l}
\text{minimize } 2x_1 + x_2 \\
\text{subject to } x_1 + x_2 \geq 1 \\
-x_1 \geq -2 \\
-x_1 + x_2 \geq -2 \\
x_1 \geq 0 \\
x_2 \geq 0
\end{array}\]

The optimal solution is \(x^* = (0 \ 1 \ 1 \ 0 \ 1)^T\) with \(\lambda^* = (1 \ 1 \ 0 \ 0 \ 0)^T\) and a final working set of \(\mathcal{W}^* = \{1, 4\}\).

We can "scatter" \(\lambda^*_i\) to get \(\lambda^* = (1 \ 0 \ 0 \ 1 \ 0)^T\).

\(\mathcal{W}^*\) implies constraints #1 and #4 are active at the solution and for \(i \notin \mathcal{W}^*, \lambda_i = 0\).

The constraint matrix is \(5 \times 2\). All-inequality simplex will solve 2 \times 2 systems.

Primal-Dual Conversion
“Taking the dual” involves three steps:

1. Convert the given LP to generic all-inequality or standard form for the primal.

2. “Take the dual”. Define the dual problem in terms of the generic quantities.


We consider a schematic representation of the problem data:

The primal in all-inequality form:
\[
\begin{align*}
& \text{minimize} & & c^T x \\
& \text{subject to} & & Ax \geq b
\end{align*}
\]

The primal in standard form:
\[
\begin{align*}
& \text{minimize} & & c^T x \\
& \text{subject to} & & Ax = b, \quad x \geq 0
\end{align*}
\]

The dual of the all-inequality primal problem:
\[
\begin{align*}
& \text{minimize} & & (-b)^T \lambda \\
& \text{subject to} & & (-A)^T \lambda = (-c) \\
& & & \lambda \geq 0
\end{align*}
\]

The dual of the standard-form primal problem:
\[
\begin{align*}
& \text{minimize} & & (-b)^T \pi \\
& \text{subject to} & & (-A)^T \pi \geq (-c)
\end{align*}
\]

Given data \( A, b \) and \( c \):

\( \text{IF}(A, b, c) \) denotes an LP in all-inequality form:
\[
\begin{align*}
& \text{minimize} & & c^T x \\
& \text{subject to} & & Ax \geq b
\end{align*}
\]

\( \text{SF}(A, b, c) \) denotes an LP in standard form:
\[
\begin{align*}
& \text{minimize} & & c^T x \\
& \text{subject to} & & Ax = b \quad \text{and} \quad x \geq 0
\end{align*}
\]
1. **Convert.** Transform an arbitrary linear program into one of the two primal templates \( \text{IF} (A, b, c) \) or \( \text{SF} (A, b, c) \).

2. **Take the dual.**
   The dual of the primal \( \text{IF} (A, b, c) \) is \( \text{SF} (-A^T, -c, -b) \)
   The dual of the primal \( \text{SF} (A, b, c) \) is \( \text{IF} (-A^T, -c, -b) \)

   These rules may be memorized using the following mnemonic:

   \[
   \begin{bmatrix}
   c^T \\
   A \\
   b
   \end{bmatrix}
   \quad \text{transpose and negate} \quad
   \begin{bmatrix}
   -b^T \\
   -A^T \\
   -c
   \end{bmatrix}
   \]

   where the first row designates the objective vector and the second row designates the general constraint matrix and its associated right-hand side.

3. **Simplify.** Make the appearance of the dual linear program as simple as possible.

**Example 1:**
Consider the problem

\[
\begin{array}{c}
\text{minimize} \\
\begin{bmatrix}
0 \\
G \\
I_n
\end{bmatrix}
\end{array}
\begin{bmatrix}
w \\
f
0
\end{bmatrix}
\begin{array}{c}
\text{subject to} \\
Gw \geq f, \quad w \geq 0
\end{array}
\]

**Convert.** We choose to write the primal in all-inequality form

\[
\begin{bmatrix}
G \\
I_n
\end{bmatrix}
\begin{bmatrix}
w \\
f
0
\end{bmatrix}
\geq 0
\]

This problem has data:

\[
\begin{bmatrix}
\begin{bmatrix}
d^T \\
G \\
I_n
\end{bmatrix} \\
\begin{bmatrix}
f \\
0
\end{bmatrix}
\end{bmatrix}
\]

**Take the dual.**

\[
\begin{bmatrix}
\begin{bmatrix}
d^T \\
G \\
I_n
\end{bmatrix} \\
\begin{bmatrix}
f \\
0
\end{bmatrix}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-f^T \\
-G^T \\
-I_n
\end{bmatrix}
\begin{bmatrix}
y \\
f
0
\end{bmatrix}
\]

The dual form is in standard form:

\[
\begin{array}{c}
\text{minimize} \\
\begin{bmatrix}
-f^T \\
-G^T \\
-I_n
\end{bmatrix}
\begin{bmatrix}
y \\
f
0
\end{bmatrix}
\begin{array}{c}
\text{subject to} \\
(G^T I_n) y = d, \quad y \geq 0
\end{array}
\end{array}
\]

**Simplify.** Reversing the sign of the equality constraints gives

\[
\begin{array}{c}
\text{minimize} \\
\begin{bmatrix}
-f^T \\
-G^T \\
-I_n
\end{bmatrix}
\begin{bmatrix}
y \\
f
0
\end{bmatrix}
\begin{array}{c}
\text{subject to} \\
(G^T I_n) y = d, \quad y \geq 0
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{minimize} \\
\begin{bmatrix}
-f^T \\
-G^T \\
-I_n
\end{bmatrix}
\begin{bmatrix}
y \\
f
0
\end{bmatrix}
\begin{array}{c}
\text{subject to} \\
(G^T I_n) y = d, \quad y \geq 0
\end{array}
\end{array}
\]
If we partition \( y \) as
\[
y = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{with} \quad u \in \mathbb{R}^m, \quad v \in \mathbb{R}^n
\]
then the problem may be written as
\[
\begin{align*}
\text{minimize} & \quad -f^T u \\
\text{subject to} & \quad G^T u + v = d, \quad \text{with} \quad u \geq 0 \quad \text{and} \quad v \geq 0
\end{align*}
\]

Next we reverse the sign of the objective, to give
\[
\begin{align*}
\text{maximize} & \quad f^T u \\
\text{subject to} & \quad G^T u + v = d, \quad \text{with} \quad u \geq 0 \quad \text{and} \quad v \geq 0
\end{align*}
\]
The elements of \( v \) may interpreted as slack variables for the constraints \( G^T u \leq d \), giving
\[
\begin{align*}
\text{maximize} & \quad f^T u \\
\text{subject to} & \quad G^T u \leq d, \quad u \geq 0
\end{align*}
\]

The final primal to dual transformation is:
\[
\begin{align*}
(P) : \quad & \text{minimize} \quad d^T w \\
& \text{subject to} \quad Gw \geq f, \quad w \geq 0
\end{align*}
\]
\[
\begin{align*}
(D) : \quad & \text{maximize} \quad f^T u \\
& \text{subject to} \quad G^T u \leq d, \quad u \geq 0
\end{align*}
\]

To get the dual we started by converting the problem to all-inequality form.

We get the same result if we start by converting the problem to standard form.

**Example 2:**
\[
\begin{align*}
\text{minimize} & \quad d^T w \\
\text{subject to} & \quad Gw \geq f, \quad w \geq 0
\end{align*}
\]

**Convert.** Write the primal in standard form
\[
Gw - s = (G - I_m) \begin{pmatrix} w \\ s \end{pmatrix} = f, \quad \begin{pmatrix} w \\ s \end{pmatrix} \geq 0
\]
This problem has data:
\[
\begin{bmatrix}
    d^T & 0 \\
    G & -I_m
\end{bmatrix}_{sf}
\]
Take the dual. The dual is in all-inequality form, with data:

\[
\begin{bmatrix}
-f^T \\
-G^T \\
-(-I_m) \\
0
\end{bmatrix}
\]

The all-inequality-form problem is

\[
\begin{array}{l}
\text{minimize} \quad (-f)^T y \\
\text{subject to} \quad (-G^T) y \geq -d, \quad y \geq 0
\end{array}
\]

Simplify. If we split the constraints, then

\[
\begin{array}{l}
\text{minimize} \quad (-f)^T y \\
\text{subject to} \quad (-G^T) y \geq -d, \quad y \geq 0
\end{array}
\]

From the preceding slide:

\[
\begin{array}{l}
\text{minimize} \quad (-f)^T y \\
\text{subject to} \quad (-G^T) y \geq -d, \quad y \geq 0
\end{array}
\]

Multiplying the objective and constraints by \(-1\) gives

\[
\begin{array}{l}
\text{maximize} \quad f^T y \\
\text{subject to} \quad G^T y \leq d, \quad y \geq 0
\end{array}
\]

which is the dual we found by starting with all-inequality form.