Optimization Tools and Resources

- Modeling languages: AMPL, GAMS, JuMP (in Julia)
  ```
  using JuMP
  m = Model()
  @variable(m, 0 <= x <= 2 )
  @variable(m, 0 <= y <= 30 )
  @objective(m, Max, 5x + 3*y )
  @constraint(m, 1x + 5y <= 3.0 )
  status = solve(m)
  ```
- Optimization solvers for LP (and more!): CPLEX, Fico-Xpress, Gurobi, Ipopt, Knitro, Mosek, SNOPT, and many more...
- Other areas of optimization: convex, integer, quadratic, nonlinear, stochastic, ...
- Online optimization resource: www.neos-server.org
  - offers many different optimization solvers for free online
  - problems have to be written in particular format
Zero-Sum Games (An economics application)

A zero-sum game is a two player, where one player’s loss is the other’s gain. The following is a payoff matrix for “rock, paper, scissors”:

\[
\begin{pmatrix}
\text{rock} & \text{paper} & \text{scissors} \\
\text{rock} & 0 & -1 & 1 \\
\text{paper} & 1 & 0 & -1 \\
\text{scissors} & -1 & 1 & 0 \\
\end{pmatrix} = A
\]

- \(a_{ij}\) is the payoff that row player has to pay the column player
- rows represent strategies \(i \in \{1, \ldots, m\}\) for the row player
- columns represent strategies \(j \in \{1, \ldots, n\}\) for the column player

Let \(y_i\) and \(x_j\) be the probabilities of the row and column player picking strategy \(i\) and \(j\), respectively. \(\gamma\) and \(\delta\) are the expected utilities for each player.

Column player’s strategy is:

\[
\begin{align*}
\text{maximize} & \quad \gamma \\
\text{subject to} & \quad Ax \geq \gamma e \\
& \quad e^T x = 1 \\
& \quad x \geq 0 \\
\end{align*}
\]

Row player’s strategy is:

\[
\begin{align*}
\text{minimize} & \quad \delta \\
\text{subject to} & \quad A^T y \leq \delta e \\
& \quad e^T y = 1 \\
& \quad y \geq 0 \\
\end{align*}
\]

These are actually dual problems of each other!
Linear Classifier (learning)

Suppose we have some data that has been categorized into two distinct groups. We want to compute a line $f(x) = v^T x - \gamma$ that can separate this data into their two sets, so either $f(x) \geq 0$ or $f(x) < 0$.

The idea is that if we can find this line with the given data, then we can use this line to classify other data. What’s the best line?

Assume we have set $D \subseteq \mathbb{R}^n$ of points and they are classified for us and for every $p^i \in D$, $y^i = \pm 1$ indicating the point’s classification.

Can we find the line that separates the two groups?

\[
\begin{align*}
\text{minimize} & \
\delta \\
\text{subject to} & \
y^i(v^T p^i + \gamma) + \delta \geq 1, \\ & \delta \geq 0
\end{align*}
\]

If we find a solution with $\delta = 0$, then we have a line $f(x) = v^T x + \gamma$ that correctly classifies every point in $D$. 

Linear Regression

Given data points \((x_i, y_i)\), can we find a linear function that best fits the data (i.e. \(f(x_i) = \alpha x_i + \beta \approx y_i\) for all \(i\))?

In other words, we want to minimize the error

\[
\sum_i |f(x_i) - y_i| = \sum_i |\alpha x_i + \beta - y_i|
\]

This is equivalent to the (almost) standard-form LP:

\[
\begin{align*}
\text{minimize} & \quad e^T (z^+ + z^-) \\
\text{subject to} & \quad \alpha x_i + \beta - y_i = z_i^+ - z_i^- \text{ for all } i \\
& \quad z^+, z^- \geq 0
\end{align*}
\]

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Final Exam

- Final examination is on Friday March 22 from 11:30am – 2:30pm
- The exam will be in Pepper Canyon 106 or Center 113
- You will be assigned a seat via tritoned. I will send an email when this is available. Please look up your seat and the room you are assigned to before you arrive for the exam
- Bring your ID
Class Summary

• The formulation of simple linear programs
  • Diet optimization; juice, beer production; etc.
  • The graphical method (geometry of constraint and objective hyperplanes)
• Linear programs with constraints in all-inequality form and standard form
  • Optimality conditions; determining whether a point is optimal or not
  • Definition of a vertex
  • Converting between each problem format
• The simplex method for constraints in all-inequality form and standard form
  • Know how to apply each version to a problem
  • Know when the algorithms stop
  • Understand the conditions required to start the methods
  • Effects of degeneracy (in all-inequality simplex)

• Finding a feasible point (phase-1 LP)
  • Write down the appropriate phase-1 LP for standard-form and all-inequality problems
  • Define an appropriate starting vertex for the phase-1 LP
• Primal and dual forms of a linear program
  • Converting from all-inequality primal to standard-form dual
  • Converting from standard-form primal to all-inequality dual

\[
\begin{bmatrix}
    c^T \\
    A & b
\end{bmatrix}_{\text{SF | IF}} \rightarrow \begin{bmatrix}
    -b^T \\
    -A^T & -c
\end{bmatrix}_{\text{IF | SF}} \text{ transpose and negate}
\]

Remember the nonnegativity constraints if you’re converting to SF form!
• Proofs?
  • Why is Dantzig’s rule “better”?
  • HW 4.1, 4.2, 4.4, 5.1, 5.4