Last lecture

- LPs in all-inequality form
- Definitions: feasible, infeasible, strictly feasible, active
- Normal vector of a constraint
Distance of a point to a hyperplane

Result

Given $x_0 \in \mathbb{R}^n$ and a hyperplane $a^T x = b$, the quantity

$$\frac{|a^T x_0 - b|}{\|a\|}$$

measures the perpendicular distance of $x_0$ to $a^T x = b$.

Navigating a Hyperspace

Definition

The value or residual of a constraint $a^T x \geq b$ at a point $\bar{x}$ is $r(\bar{x}) \equiv a^T \bar{x} - b$.

At an arbitrary point $\bar{x}$

$$r(\bar{x}) = \begin{cases} 
\geq 0, & \text{if } a^T \bar{x} \geq b \text{ is satisfied (feasible) at } \bar{x} \\
= 0, & \text{if } a^T \bar{x} \geq b \text{ is active at } \bar{x} \\
< 0, & \text{if } a^T \bar{x} \geq b \text{ is violated (infeasible) at } \bar{x}
\end{cases}$$
• The \textit{sign} of \( r(\bar{x}) \) tells us which side of the hyperplane \( a^T x = b \) \( \bar{x} \) lies.

• If \( r(\bar{x}) = 0 \) then \( \bar{x} \) lies \textit{on} the hyperplane, i.e., \( a^T x \geq b \) is \textit{active} at \( \bar{x} \).

• \( \bar{x} \) is distance \( \frac{|a^T \bar{x} - b|}{\|a\|} = \frac{|r(\bar{x})|}{\|a\|} \) from the hyperplane.

Example: \(-6x_1 + x_2 - 2x_3 \leq -1\) and \( \bar{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T \)
Is \( \bar{x} \) feasible w.r.t. the constraint? How far is \( \bar{x} \) from the hyperplane?

Writing the constraint in generic form gives
\[
6x_1 - x_2 + 2x_3 \geq 1, \quad \text{i.e.,} \quad a = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} \quad \text{with} \quad b = 1
\]

At \( \bar{x} \), we have
\[
r(\bar{x}) = a^T \bar{x} - b = \begin{pmatrix} 6 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1 = 7 - 1 = 6 > 0
\]
\( \Rightarrow \) \( \bar{x} \) is strictly feasible.
\( \Rightarrow \) \( \bar{x} \) is distance \( \frac{6}{\sqrt{41}} \approx 0.937 \) from the hyperplane.
What about $\bar{x} = (1 \ -1 \ -3)^T$?

$$r(\bar{x}) = a^T\bar{x} - b = (6 \ -1 \ 2) \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} - 1 = 0$$

$\Rightarrow$ the constraint is active at $\bar{x}$.

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**Moving in hyperspace**

Move along a path starting at $\bar{x}$

Parameterize by a scalar $\alpha$ ($\alpha \geq 0$)

Every point $x$ on the path is written in the form $x = x(\alpha)$.

The value of $\alpha$ tells us how far we are along the path.
\[ x(\alpha) = \begin{pmatrix} x_1(\alpha) \\ x_2(\alpha) \\ \vdots \\ x_n(\alpha) \end{pmatrix}, \text{ with } x(0) = \bar{x} \]

For example, for \( 0 \leq \alpha \leq 2\pi \), define the path

\[ x(\alpha) = \begin{pmatrix} x_1(\alpha) \\ x_2(\alpha) \end{pmatrix} = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \text{ with } \bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

The same points are defined by the path:

\[ x(\alpha) = \begin{pmatrix} \sin 3\alpha \\ \cos 3\alpha \end{pmatrix}, \text{ with } \bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } 0 \leq \alpha \leq \frac{2}{3}\pi \]
We can define the linear path:

\[ x(\alpha) = \bar{x} + \alpha p \]

where \( p \) is a nonzero direction vector and \( \alpha \) is the step length.

**Example:**

Given \( \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and the direction \( p = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

the linear path is

\[ x(\alpha) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 + \alpha \end{pmatrix} \text{ for all } \alpha \geq 0 \]
Now consider an inequality constraint $a^T x \geq b$. Let’s consider the residual along the direction $p$.

$$
\begin{align*}
r(\bar{x} + \alpha p) &= a^T (\bar{x} + \alpha p) - b \\
&= a^T \bar{x} + \alpha a^T p - b \\
&= (a^T \bar{x} - b) + \alpha a^T p \\
&= r(\bar{x}) + \alpha a^T p
\end{align*}
$$

$\Rightarrow$ $r(\bar{x} + \alpha p)$ is a linear function of $\alpha$, with

$$
\frac{d}{d\alpha} r(\bar{x} + \alpha p) \bigg|_{\alpha=0} = a^T p
$$

$\Rightarrow$ $r$ changes at the constant rate $a^T p$.

$\Rightarrow$ $r$ is

$$
\begin{align*}
\text{decreasing if } & a^T p < 0 \\
\text{constant if } & a^T p = 0 \\
\text{increasing if } & a^T p > 0
\end{align*}
$$

Since $|r|$ reflects the distance to the constraint, the sign of $a^T p$ tells us if we are moving away from, or towards the constraint boundary.
Suppose that $\bar{x}$ is feasible.

Let $p^{(1)}$, $p^{(2)}$, and $p^{(3)}$ be directions such that

\[
\begin{align*}
    a^T p^{(1)} &< 0 \\
    a^T p^{(2)} &= 0 \\
    a^T p^{(3)} &> 0
\end{align*}
\]

Then:

\[
\begin{align*}
    a^T p^{(1)} < 0 &\implies p^{(1)} \text{ points towards the constraint boundary} \\
    a^T p^{(2)} = 0 &\implies p^{(2)} \text{ is parallel to the constraint boundary} \\
    a^T p^{(3)} > 0 &\implies p^{(3)} \text{ points away from the constraint boundary}
\end{align*}
\]

Summary

Given the hyperplane $a^T x \geq b$, a feasible $\bar{x}$ and any direction $p \neq 0$, a step $\alpha > 0$ along $p$

- moves towards the infeasible half space if $a^T p < 0$ and
- moves away from the infeasible half space if $a^T p > 0$.
- If $a^T p = 0$, the constraint remains constant for any move along $p$. 

Summary

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![](image.png)

$a$ points into the feasible region, away from the boundary