Math 171A: Linear Programming

Lecture 3
More on Linear Constraints

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Distance of a point to a hyperplane

Result
Given \( x_0 \in \mathbb{R}^n \) and a hyperplane \( a^T x = b \), the quantity
\[
\frac{|a^T x_0 - b|}{\|a\|}
\]
measures the perpendicular distance of \( x_0 \) to \( a^T x = b \).

Navigating a Hyperspace

Definition
The value or residual of a constraint \( a^T x \geq b \) at a point \( \bar{x} \) is
\[
r(\bar{x}) \triangleq a^T \bar{x} - b.
\]
At an arbitrary point \( \bar{x} \)
\[
r(\bar{x}) \begin{cases} 
\geq 0, & \text{if } a^T x \geq b \text{ is satisfied (feasible) at } \bar{x} \\
= 0, & \text{if } a^T x \geq b \text{ is active at } \bar{x} \\
< 0, & \text{if } a^T x \geq b \text{ is violated (infeasible) at } \bar{x}
\end{cases}
\]
• The sign of $r(\bar{x})$ tells us which side of the hyperplane $a^T x = b$ $\bar{x}$ lies.
• If $r(\bar{x}) = 0$ then $\bar{x}$ lies on the hyperplane, i.e., $a^T x \geq b$ is active at $\bar{x}$.
• $\bar{x}$ is distance $\frac{|a^T \bar{x} - b|}{\|a\|} = \frac{|r(\bar{x})|}{\|a\|}$ from the hyperplane.

Example: $-6x_1 + x_2 - 2x_3 \leq -1$ and $\bar{x} = (1 \ 1 \ 1)^T$
Is $\bar{x}$ feasible w.r.t. the constraint? How far is $\bar{x}$ from the hyperplane?
Writing the constraint in generic form gives
$$6x_1 - x_2 + 2x_3 \geq 1,$$ i.e., $a = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$ with $b = 1$
At $\bar{x}$, we have
$$r(\bar{x}) = a^T \bar{x} - b = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - 1 = 7 - 1 = 6 > 0$$
$\Rightarrow$ $\bar{x}$ is strictly feasible.
$\Rightarrow$ $\bar{x}$ is distance $\frac{6}{\sqrt{41}} \approx 0.937$ from the hyperplane.

Moving in hyperspace

Move along a path starting at $\bar{x}$

What about $\bar{x} = (1 \ -1 \ -3)^T$?

$$r(\bar{x}) = a^T \bar{x} - b = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} - 1 = 0$$
$\Rightarrow$ the constraint is active at $\bar{x}$.

Parameterize by a scalar $\alpha$ ($\alpha \geq 0$)
Every point $x$ on the path is written in the form $x = x(\alpha)$.
The value of $\alpha$ tells us how far we are along the path.
\[ x(\alpha) = \begin{pmatrix} x_1(\alpha) \\ x_2(\alpha) \\ \vdots \\ x_n(\alpha) \end{pmatrix}, \text{ with } x(0) = \bar{x} \]

For example, for \(0 \leq \alpha \leq 2\pi\), define the path

\[ x(\alpha) = \begin{pmatrix} x_1(\alpha) \\ x_2(\alpha) \end{pmatrix} = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \text{ with } \bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

The same points are defined by the path:

\[ x(\alpha) = \begin{pmatrix} \sin 3\alpha \\ \cos 3\alpha \end{pmatrix}, \text{ with } \bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } 0 \leq \alpha \leq \frac{2\pi}{3} \]

Example:
Given \(\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and the direction \(p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\), the linear path is

\[ x(\alpha) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 + \alpha \end{pmatrix} \text{ for all } \alpha \geq 0 \]
Now consider an inequality constraint $a^T x \geq b$. Let’s consider the residual along the direction $p$.

$$r(\bar{x} + \alpha p) = a^T(\bar{x} + \alpha p) - b$$

$$= a^T\bar{x} + \alpha a^T p - b$$

$$= (a^T\bar{x} - b) + \alpha a^T p$$

$$= r(\bar{x}) + \alpha a^T p$$

$\Rightarrow r(\bar{x} + \alpha p)$ is a linear function of $\alpha$, with

$$\frac{d}{d\alpha} r(\bar{x} + \alpha p) \bigg|_{\alpha=0} = a^T p$$

$\Rightarrow r$ changes at the constant rate $a^T p$.

Suppose that $\bar{x}$ is feasible.

Let $p(1)$, $p(2)$, and $p(3)$ be directions such that

$$a^T p(1) < 0$$

$$a^T p(2) = 0$$

$$a^T p(3) > 0$$

Then:

$$a^T p(1) < 0 \implies p(1) \text{ points towards the constraint boundary}$$

$$a^T p(2) = 0 \implies p(2) \text{ is parallel to the constraint boundary}$$

$$a^T p(3) > 0 \implies p(3) \text{ points away from the constraint boundary}$$

$\Rightarrow r$ is

$$\begin{align*}
\text{decreasing if } & a^T p < 0 \\
\text{constant if } & a^T p = 0 \\
\text{increasing if } & a^T p > 0
\end{align*}$$

Since $|r|$ reflects the distance to the constraint, the sign of $a^T p$ tells us if we are moving away from, or towards, the constraint boundary.

Summary

Given the hyperplane $a^T x \geq b$, a feasible $\bar{x}$ and any direction $p \neq 0$, a step $\alpha > 0$ along $p$

- moves towards the infeasible half space if $a^T p < 0$ and

- moves away from the infeasible half space if $a^T p > 0$.

- If $a^T p = 0$, the constraint remains constant for any move along $p$. 
Summary

Given the hyperplane $a^T x \geq b$, a feasible $\bar{x}$ and any direction $p \neq 0$, a step $\alpha > 0$ along $p$

- moves towards the infeasible half space if $a^T p < 0$ and
- moves away from the infeasible half space if $a^T p > 0$.
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a points into the feasible region, away from the boundary