The graphical method

1. Graph the feasible region $\mathbb{F}$

2. Find a corner point of $\mathbb{F}$ and a level curve $\{x : c^T x = z_0\}$ such that
   
   (a) $z_0$ is as small as possible
   
   (b) the level curve passes through the corner point

3. Select two hyperplanes that pass through the corner point and solve for $x$
Example 1:

minimize \( \begin{bmatrix} 2x_1 + x_2 \\
subject to \begin{align*}
x_1 + x_2 &\geq 1 \\
x_1 &\geq 0 \\
-x_1 &\geq -2 \\
x_2 &\geq 0 \\
x_1 + 2x_2 &\geq 1
\end{align*}
\)

minimize \( c^T x \)
subject to \( Ax \geq b \)

\( c = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)

\( A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \)

\( b = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \)
Graph the constraints

minimize \[ 2x_1 + x_2 \]
subject to
\[ x_1 + x_2 \geq 1 \]
\[ x_1 \geq 0 \]
\[ -x_1 \geq -2 \]
\[ x_2 \geq 0 \]
\[ x_1 + 2x_2 \geq 1 \]
Consider one level curve of the objective. Determine the direction that decreases the objective and find the corner point corresponding to the best level curve.

\[
\begin{align*}
\text{minimize} & \quad 2x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 \geq 1 \\
& \quad x_1 \geq 0 \\
& \quad -x_1 \geq -2 \\
& \quad x_2 \geq 0 \\
& \quad x_1 + 2x_2 \geq 1
\end{align*}
\]
$c^T x = 1$

$x^*$
The optimal corner point lies at the intersection of the hyperplanes

\[ \mathcal{H}_1 = \{ x : x_1 + x_2 = 1 \} \]
\[ \mathcal{H}_2 = \{ x : x_1 = 0 \} \]

\( \implies \) \( x^* \) is the optimal solution and satisfies the equations

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

The optimal objective value is

\[ c^T x^* = (2 \ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \]
Example 2:

\[
\begin{align*}
\text{minimize } & \quad c^T x \\
\text{subject to } & \quad Ax \geq b
\end{align*}
\]

with

\[
c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \text{and } b = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}
\]

Only the objective has changed from Example 1.
In this case, \( c \) is parallel to \( a_1 \) and there are \textit{infinitely many} solutions (all with the same optimal objective value)

Nevertheless, there are two optimal corner points, at

\[
x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad x^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
Example 3:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\]

with

\[
c = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}
\]

Again, only the objective has been changed.
In this case, we can reduce $z_0 = c^T x$ to minus infinity and never encounter a corner point.

The objective function is *unbounded below* in the feasible region.

We say that the “problem is unbounded”.
How do we solve the problem when \( n > 2 \)?

Assume that \( m \geq n \).

A bounded solution always lies at a corner point (well, almost always).

\[ \Rightarrow \text{ evaluate } \ell(x) \text{ at every corner point.} \]

A corner point must lie on \( n \) hyperplanes.

\[ \Rightarrow \text{ find a corner point by gathering } n \text{ hyperplanes from } A \text{ and } b. \]
If $\bar{A}$ is nonsingular and the solution of $\bar{A}\bar{x} = \bar{b}$ is feasible, then $\bar{x}$ is a corner point.
1. Define $\ell^* = +\infty$

2. Gather a subset of $n$ hyperplanes from the $m$ rows of $A$.
   
   (a) if $\bar{A}$ is singular, continue at Step 2.
   
   (b) Solve $\bar{A}\bar{x} = \bar{b}$

   (c) If $\bar{x}$ is infeasible, continue at Step 2.

   (d) $\ell^* = \min\{\ell^*, c^T\bar{x}\}$

   (e) If all subsets of hyperplanes have been examined, stop. Otherwise, continue at Step 2.
How efficient is this method?

Recall that there are at most \( \binom{m}{n} = \frac{m!}{n!(m-n)!} \) corner points.

Case 1: \( m = 5, \ n = 2 \):

\[
\binom{5}{2} = \frac{5!}{2! \ 3!} = 10 \text{ corner points (at most)}
\]

Case 2: \( m = 20, \ n = 10 \):

\[
\binom{20}{10} = \frac{20!}{10! \ 10!} = 184756 \text{ corner points (at most)}
\]

Assuming \( n^3 \) operations to solve for \( x \)

\( \Rightarrow \approx 1.8 \text{ billion calculations!} \)
Summary: basic properties of an LP

- An LP is either *infeasible*, *unbounded* or has an *optimal solution*.

- An *optimal solution* always lies on the boundary of the feasible region.

- Every point on the boundary of the feasible region satisfies a *linear system of equations* that is either square, underdetermined or overdetermined.
Aim of the class

(I) Given a boundary point $\bar{x}$, can we determine if $\bar{x}$ is optimal without the need to evaluate $c^T x$ at every other corner point?

(II) If $\bar{x}$ is not a solution, can we determine a direction $p$ along which $c^T x$ decreases?
Properties of Linear Equations
Consider the linear equation

$$Ax = b$$

where $A$ is an $m \times n$ matrix and $b$ is an $m$-vector, or equivalently, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

We make no assumptions on the shape of $A$

$\Rightarrow$ In general, we cannot say that $x = A^{-1}b$. 
In matrix form: \( b = Ax \), where

\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m \\
\end{pmatrix}, \quad
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m1} & \cdots & a_{mn} \\
\end{pmatrix}, \quad
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{pmatrix}
\]

This is a \textit{system of linear equations}. 
Another notation: write $A$ by *columns*:

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \text{ with } a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$$\implies Ax = \sum_{j=1}^{n} a_j x_j$$

*Can we find $x$ such that $Ax = b$?*
Definition

A system of equations is *compatible* if there exists a vector \( x \) such that \( b = Ax \).

i.e., there exist \( x_1, x_2, \ldots, x_n \) such that \( b = \sum_{j=1}^{n} a_j x_j \)

i.e., \( b \) can be written as a *linear combination* of the columns of \( A \).

A compatible system may be regarded as defining an *expansion* or *representation* of \( b \) in terms of the columns of \( A \).
Example:

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \]

\[ a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \]

The system \( Ax = b \) is compatible because

\[ \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \implies x = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \]
Definition

If there is no $x$ such that $b = Ax$, then the system is said to be **incompatible**.

**Example:** If

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then there is no $x$ such that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
The columns of $A$ are \textit{linearly dependent} if there exist scalars $z_1, z_2, \ldots, z_n$ (not all zero) such that

$$0 = \sum_{j=1}^{n} a_j z_j$$

Or equivalently, there exists a vector $z \neq 0$ such that

$$A z = 0.$$
When the columns of $A$ are dependent, we can expect to define $b$ in an *infinite number of ways* using redundant information in $A$.

**Result**

If the system $Ax = b$ is compatible and the columns of $A$ are linearly dependent, then there are *infinitely many* solutions of $Ax = b$. 
Proof: Let $\bar{x}$ be a solution of $Ax = b$, i.e., $A\bar{x} = b$.

As the columns of $A$ are linearly dependent, there is a nonzero $z$ such that $Az = 0$.

Let $\alpha$ be any scalar. Then

$$A(\bar{x} + \alpha z) = A\bar{x} + \alpha Az = b + 0 = b$$

$\Rightarrow \bar{x} + \alpha z$ is a solution for any value of $\alpha$.  \[
\]

Example:

For

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \]

We have

\[ \bar{x} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \]

which implies that any vector of the form

\[ \bar{x} + \alpha z = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 - 3\alpha \\ 1 + \alpha \\ 2\alpha \end{pmatrix} \]

is a solution.
Definition

The columns of $A$ are said to be *linearly independent* if the only vector $z$ such that

$$0 = \sum_{j=1}^{n} a_j z_j = Az$$

is the vector $z = 0$. 
Result

If the system $Ax = b$ is compatible and the columns of $A$ are linearly independent, then $\bar{x}$ is the unique solution of $Ax = b$. 
Proof: Let $x$ and $y$ be solutions of $Ax = b$, i.e.,

$$b = Ax \quad \text{and} \quad b = Ay$$

Then

$$b - b = 0 = Ax - Ay = A(x - y)$$

which implies that

$$Az = 0 \quad \text{for} \quad z = x - y$$

Because the columns of $A$ are linearly independent, we have

$$Az = \sum_{j=1}^{n} a_j z_j = 0 \quad \implies \quad z_j = 0$$

so $z = x - y = 0 \implies x = y$. \qed
Example: For

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}
\]

the columns of \( A \) are linearly independent, with

\[
A\bar{x} = b \quad \text{for} \quad \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[\Rightarrow \bar{x} \text{ is the only } x \text{ such that } Ax = b.\]