Recap

**ELP (Equality-constrained LP)**

\[
\begin{align*}
\text{minimize} & \quad \ell(x) = c^T x \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

The feasible region is \( F = \{ x \in \mathbb{R}^n : Ax = b \} \).

The set of **feasible directions** is \( \{ p \in \mathbb{R}^n : p \neq 0, \ Ap = 0 \} \).

We are looking for **computationally tractable** conditions for optimality.
Recap

**Definition**

$x^*$ is a *minimizer* of ELP if

$$c^T x^* \leq c^T x \text{ for all } x \in F = \{x \in \mathbb{R}^n : Ax = b\}$$

(i.e., $\ell(x^*) \leq \ell(x)$ for all feasible $x$).

**Result**

$x^*$ is a *minimizer* of ELP *if and only if* $\ell$ does not decrease along any feasible direction $p$ starting at $x^*$.

i.e.,

$$\left. \frac{d}{d\alpha} \ell(x^* + \alpha p) \right|_{\alpha=0} \geq 0 \text{ for every } p \text{ such that } Ap = 0$$

Recall that

$$\left. \frac{d}{d\alpha} \ell(x^* + \alpha p) \right|_{\alpha=0} = c^T p$$
Result
$x^*$ is a \textit{minimizer} of ELP if and only if
\[ c^T p \geq 0 \text{ for every } p \text{ such that } A p = 0 \]

Result
$x^*$ is a \textit{minimizer} of ELP if and only if there is no \( p \) exists such that
\[ c^T p < 0 \text{ and } A p = 0 \]

Definition
A direction \( p \) such that \( c^T p < 0 \) and \( A p = 0 \) is known as a \textit{feasible descent direction}.

Result
$x^*$ is a \textit{minimizer} of ELP if and only if no feasible descent direction exists.
Result

\[ c^T p \geq 0 \text{ for every } p \text{ such that } Ap = 0 \]

if and only if

\[ c \in \text{range}(A^T) \]

Proof: (⇐)

Assume that \( c \in \text{range}(A^T) \). Then there exists a vector \( y \) such that \( c = A^T y \).

Given this information, let's consider \( c^T p \) for some \( p \) with \( Ap = 0 \):

\[
c^T p = (A^T y)^T p = y^T(Ap) = 0.
\]

Thus, \( c^T p \geq 0 \) for all \( p \) such that \( Ap = 0 \).
Proof: \(\Rightarrow\) (by contrapositive argument)

Suppose that \(c \not\in \text{range}(A^T)\). We want to show that there exists a \(p\) such that \(Ap = 0\) and \(c^T p < 0\).

We can uniquely write \(c\) as \(c = c_R + c_N\) with \(c_R \in \text{range}(A^T)\) and \(c_N \in \text{null}(A)\). Since we assume \(c \not\in \text{range}(A^T)\), then \(c_N \neq 0\).

Define \(p = -c_N\).

\[
\begin{align*}
c^T p &= -c^T c_N \\
&= -(c_R + c_N)^T c_N \\
&= -c_R^T c_N - c_N^T c_N \\
&= -c_N^T c_N \\
&= -\|c_N\|^2 < 0
\end{align*}
\]

\(\Rightarrow p = -c_N\) satisfies \(Ap = 0\) and \(c^T p < 0\). \(\blacksquare\)

c\(^T\)x has a bounded minimizer if and only if \(c\) is in range\((A^T)\).

**Definition (Lagrange multipliers)**

The components of the vector \(\lambda^*\) such that \(c = A^T \lambda^*\) are called **Lagrange multipliers**.
Result

$x^*$ is a minimizer of ELP if and only if $Ax^* = b$ and there exist Lagrange multipliers $\lambda^*$ such that $c = A^T\lambda^*$.

Note the three equivalent forms of the optimality conditions:

- $c \in \text{range}(A^T)$
- The equations $A^T\lambda = c$ are compatible
- No feasible descent direction exists

These conditions are all independent of $x$ (!)
Example 1: Consider the problem

\[
\begin{align*}
\text{minimize} & \quad 2x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1
\end{align*}
\]

We have \( A = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} \frac{2}{2} \end{pmatrix} \)

At the point \( \bar{x} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \), we have \( A\bar{x} = b \Rightarrow \bar{x} \) is feasible.

The Lagrange multiplier equations are

\[
A^T\bar{\lambda} = c \quad \Rightarrow \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bar{\lambda} = \begin{pmatrix} \frac{2}{2} \end{pmatrix} \quad \text{for} \quad \bar{\lambda} = 2
\]

\( \Rightarrow \bar{x} \) is optimal.
Result

Let \( x^* \) be a minimizer of ELP, i.e.,

\[
Ax^* = b \quad \text{and} \quad c = A^T \lambda^*
\]

If \( \tilde{x} \) is any feasible point, then \( \ell(\tilde{x}) = \ell(x^*) \).

Proof: We have

\[
\ell(\tilde{x}) = c^T \tilde{x} = (A^T \lambda^*)^T \tilde{x} = \lambda^* A \tilde{x} = \lambda^* b
\]

\[
= \lambda^* A x^* = c^T x^* = \ell(x^*)
\]

If \( c \in \text{range}(A^T) \) then **every feasible point is optimal.** 

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**Example 2:** Consider the problem

\[
\text{minimize} \quad x_1 + 2x_2 \quad \text{subject to} \quad x_1 + x_2 = 1
\]

We have \( A = \begin{pmatrix} 1 & 1 \end{pmatrix} \), \( b = (1) \), and \( c = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)

Consider the feasible point

\[
\tilde{x} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}
\]

\( c \not\in \text{range}(A^T) \implies \) there is no \( \lambda \) such that \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \lambda = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)
Write
\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
= \begin{pmatrix}
  \frac{3}{2} \\
  \frac{3}{2}
\end{pmatrix} + \begin{pmatrix}
  -\frac{1}{2} \\
  \frac{1}{2}
\end{pmatrix}
\]
with \(c_R \in \text{range}(A^T), \quad c_N \in \text{null}(A)\)

Define
\[
p = -c_N = -\begin{pmatrix}
  -\frac{1}{2} \\
  \frac{1}{2}
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{2} \\
  -\frac{1}{2}
\end{pmatrix}
\]
and note that
\[
Ap = \begin{pmatrix}
  1 & 1
\end{pmatrix} \begin{pmatrix}
  \frac{1}{2} \\
  -\frac{1}{2}
\end{pmatrix} = 0
\]
\[
c^T p = \begin{pmatrix}
  1 & 2
\end{pmatrix} \begin{pmatrix}
  \frac{1}{2} \\
  -\frac{1}{2}
\end{pmatrix} = -\frac{1}{2} < 0
\]
Result (Optimality Conditions for ELP)

Consider minimizing $\ell(x) = c^T x$ subject to $Ax = b$.

(a) If $Ax = b$ is incompatible, no solution exists;

(b) If $Ax = b$ is compatible and $c$ is not in the range of $A^T$, the objective function is unbounded below at feasible points;

(c) If $Ax = b$ is compatible and $c$ is in the range of $A^T$ (so that $c = A^T \lambda^*$ for some $\lambda^*$), then:
   (i) $\ell^*$, the optimal value of $\ell$, is finite and unique;
   (ii) Every feasible point is a minimizer $x^*$;
   (iii) $x^*$ is unique if and only if the columns of $A$ are linearly independent;
   (iv) $\lambda^*$ is unique if and only if the rows of $A$ are linearly independent.

Now (finally!) we consider a general linear program.

LP minimize $\quad c^T x$
subject to $\quad Ax \geq b$

with $A$ an $m \times n$ matrix, $b$ an $m$-vector and $c$ an $n$-vector.

The feasible region is

$$F = \{ x : Ax \geq b \}$$
Aim of the class

(I) Given a feasible point $\bar{x}$, can we determine if $\bar{x}$ is optimal without the need to evaluate $c^T x$ at every other corner point?

(II) If $\bar{x}$ is not optimal, can we determine a direction $p$ such that $\bar{x} + \alpha p$ is feasible and $c^T x$ decreases along $p$?

Result

A feasible point $x^*$ is a minimizer of an LP if and only if

$$\ell(x^*) \leq \ell(x^* + \alpha p) \text{ for all } \alpha \geq 0 \text{ and all feasible directions } p$$

What are the feasible directions for $Ax \geq b$?
The value or residual of the constraints $Ax \geq b$ at $\bar{x}$ is

$$r(\bar{x}) = A\bar{x} - b = \begin{pmatrix}
a_1^T\bar{x} - b_1 \\
a_2^T\bar{x} - b_2 \\
\vdots \\
a_m^T\bar{x} - b_m
\end{pmatrix}$$

At an arbitrary point $\bar{x}$

$$r_i(\bar{x}) = \begin{cases} 
\geq 0, & \text{if } a_i^T\bar{x} \geq b_i \text{ is satisfied (feasible) at } \bar{x} \\
0, & \text{if } a_i^T\bar{x} = b_i \text{ is active at } \bar{x} \\
< 0, & \text{if } a_i^T\bar{x} < b_i \text{ is violated (infeasible) at } \bar{x}
\end{cases}$$

Definition (Feasible direction for a single constraint)

Consider a constraint $a_i^T\bar{x} \geq b_i$. Given a feasible $\bar{x}$ (i.e., $r_i(\bar{x}) \geq 0$), $p \neq 0$ is a feasible direction at $\bar{x}$ if there exists a step $\sigma_i > 0$ such that $\bar{x} + \alpha p$ is feasible for all $\alpha$ such that $0 < \alpha \leq \sigma_i$.

This is the same as $r_i(\bar{x} + \alpha p) \geq 0$ for all $0 < \alpha \leq \sigma_i$, i.e., $\bar{x} + \alpha p$ is feasible for all positive steps shorter than $\sigma_i$.

First, we focus on the definition of $p$. Later, we will consider $\sigma_i$. 

Notes

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**Definition (Feasible direction for a set of constraints)**

Given a feasible \( \bar{x} \), a nonzero \( p \) is a **feasible direction** for a set of constraints \( Ax \geq b \) if there exists a step \( \sigma > 0 \) such that

\[
A(\bar{x} + \alpha p) \geq b \quad \text{for all} \quad 0 < \alpha \leq \sigma.
\]

Equivalently, \( r(\bar{x} + \alpha p) \geq 0 \) for all \( 0 < \alpha \leq \sigma \).

The set of feasible directions at \( \bar{x} \) depends on the **location** of \( \bar{x} \).
In particular, it depends on whether or not \( \bar{x} \) lies on \( a_i^T x = b_i \).