• No calculators. Please turn off all electronics.
• Show your work and justify all of your answers.
• Clearly indicate the problem you are solving.
• Please have your ID ready for inspection.
Question 1. (25 pts) Consider the linear program:

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 - 2x_3 \\
\text{subject to} & \quad -x_1 - 2x_3 \geq -2 \\
& \quad x_1 + 2x_2 + 3x_3 \geq 2 \\
& \quad 2x_2 - x_3 \geq 1 \\
& \quad x_1 \geq 1 \\
& \quad x_2 \geq 2 \\
& \quad x_3 \geq 0.
\end{align*}
\]

Write the problem in the form \( \min c^T x \) subject to \( Ax \geq b \). Given the starting vertex \( x_0 = (1 \ 2 \ 0)^T \), solve this LP using the simplex method. Show your work. Be sure to write down the working set and the Lagrange multipliers at each iteration of the simplex method. Verify that each simplex vertex is feasible and that \( c^T x_k \geq c^T x_{k+1} \).

The problem is defined as:

\[
c = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 2 & 3 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}.
\]

At \( x_0 \) we compute the residual to determine the active set:

\[
r(x_0) = Ax_0 - b = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}.\]

The active set is \( A_0 = \{4, 5, 6\} \). \( x_0 \) is a nondegenerate vertex thus the working set is equal to the active set.

\[
W_0 = \{4, 5, 6\}, \quad A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Note the active-set matrix is the identity matrix, which makes calculations very simple in the first iteration.

Step 1: Check optimality.

\[
c = A_0^T \lambda_0 \implies \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda_0 \implies \lambda_0 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.
\]

There is only one negative multiplier so \( s = 3 \) (\( w_s = 6 \)).
Step 2: Search direction.

\[ A_0 p_0 = e_s \implies \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} p_0 = e_3 \implies p_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

Step 3: Max feasible step.

\[ a_1^T p_0 = -2 \implies \sigma_1 = \frac{1}{(-2)} = \frac{1}{2} \]
\[ a_2^T p_0 = 3 \implies \sigma_2 = \infty \]
\[ a_3^T p_0 = -1 \implies \sigma_3 = \frac{3}{(-1)} = 3 \]

Our maximum feasible step is \( \alpha_0 = \sigma_1 = \frac{1}{2} \), with \( t = 1 \).

Step 4: We update our simplex iterate

\[ x_1 = x_0 + \alpha_0 p_0 \implies x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \]

and our new working set is \( W_1 = \{4, 5, 1\} \).

Step 1 (second iteration): We check optimality at \( x_1 \).

\[ c = A_1^T \lambda_1 \implies \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \lambda_1 \implies \lambda_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \geq 0. \]

Thus \( x_1 \) is optimal.

We can verify \( x_1 \) is feasible:

\[ r(x_1) = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 2 & 3 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ \frac{3}{2} \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{13}{2} \\ \frac{5}{2} \\ 1 \\ 2 \\ \frac{1}{2} \end{pmatrix} \geq 0 \]

Furthermore, the objective value at \( x_1 \) decreases:

\[ c^T x_0 = 3 \quad \text{and} \quad c^T x_1 = 2. \]
Question 2. At a feasible point $\bar{x}$, the active-constraint matrix is given by

$$A_a = \begin{pmatrix} -1 & 3 \\ 2 & 3 \\ 0 & 2 \\ -1 & -1 \\ -3 & 0 \end{pmatrix}$$

(a) (8 pts) Is $\bar{x}$ a vertex? If $\bar{x}$ is a vertex, is it degenerate or nondegenerate? Justify your answer.

The active-set matrix $A_a$ has rank 2 (the columns are clearly not multiples of each other). $\bar{x}$ is a feasible point with at least $n = 2$ constraints active. Thus, $\bar{x}$ is a vertex. Because there are $5 > n$ constraints active, $\bar{x}$ is a degenerate vertex.

(b) (12 pts) Assume the objective is defined with $c = (-1 \ 1)^T$. Determine whether $\bar{x}$ is optimal or not. If $\bar{x}$ is optimal, give an appropriate set of Lagrange multipliers. If $\bar{x}$ is not optimal, find a feasible descent direction $p$.

We apply Carathéodory’s theorem to help us determine optimality. We consider basic solutions to $c = A_a^T \lambda$ to determine if any nonnegative Lagrange multipliers exist.

Let us consider constraints #3 and #5, which are clearly linearly independent.

$$c = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix} \lambda \implies \lambda = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

The Lagrange multipliers are nonnegative, therefore, $\bar{x}$ is optimal.

Note: this is not a unique solution. There are several other choices that will give a nonnegative solution. For example, with constraints #1 and #4, you will get $\lambda = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}^T$. 
Question 3. Consider the constraints

\[-x_1 + x_2 \geq 1, \quad 2x_1 - x_2 \geq -1, \quad x_1 - x_2 \leq -2, \quad x_1 \geq 0, \quad x_2 \geq 0\]

(a) (3 pts) Verify that the point \(x_0 = 0\) is infeasible.

At \(x_0 = 0\), we violate the following constraints:

\[-x_1 + x_2 \geq 1 \implies 0 \geq 1 \quad \text{and} \quad x_1 - x_2 \leq -2 \implies 0 \leq -2.\]

The other constraints are feasible at \(x_0\).

(b) (12 pts) Formulate the phase-1 linear program. Define the phase-1 LP so that you know an initial vertex for the phase-1 constraints for which \(x_1 = x_2 = 0\).

Do NOT solve the phase-1 LP!

Based on part (a), we have 2 violated constraints that must be shifted in our phase-1 LP. We introduce the new variable \(\theta\) to represent this shift.

\[
\begin{align*}
\text{minimize} & \quad \theta \\
\text{subject to} & \quad -x_1 + x_2 + \theta \geq 1 \\
& \quad 2x_1 - x_2 \geq -1 \\
& \quad -x_1 + x_2 + \theta \geq 2 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad \theta \geq 0
\end{align*}
\]

With this phase-1 LP, the constraints \(x_1 \geq 0\) and \(x_2 \geq 0\) will be active with \(x_0 = 0\). If we set the initial shift value correctly, then one of the previously violated constraints will also be active, and we will have a vertex for the phase-1 LP.