Exercise 3.1. Let $A$ be an $m \times n$ matrix with rank($A$) = $r$.

(a) Define range($A$), null($A^T$), range($A^T$), and null($A$) and indicate which vector space $\mathbb{R}^n$ or $\mathbb{R}^m$ each is a subspace of.

- range($A$) = $\{y \in \mathbb{R}^m : y = Ax, \ x \in \mathbb{R}^n\}$
- null($A^T$) = $\{y \in \mathbb{R}^m : A^Ty = 0\}$
- range($A^T$) = $\{x \in \mathbb{R}^n : x = A^Ty, \ y \in \mathbb{R}^m\}$
- null($A$) = $\{x \in \mathbb{R}^n : Ax = 0\}$

(b) If rank($A$) = $m$, what can you say about the relative sizes of $m$ and $n$?

If $A$ has full row rank, then $m \leq n$.

(c) Similarly, if rank($A$) = $n$, what can you say about the relative sizes of $m$ and $n$?

If $A$ has full column rank, then $n \leq m$.

(d) Show that range($A^T$) $\cap$ null($A$) contains only the zero vector.

Let $y \in$ range($A^T$) $\cap$ null($A$). Then $Ay = 0$ and there exists some vector $x$ such that $y = A^Tx$.

Then $0 = Ay = A(A^Tx)$.

If we apply $x^T$ to both sides of this equation, we get

$$0 = x^TA^Tx = (A^Tx)^T(A^Tx) = \|A^Tx\|^2.$$ 

We must have $A^Tx = 0$ and therefore, $y = 0$, which shows that the only vector in the intersection of range($A^T$) and null($A$) is the zero vector.

Exercise 3.2. Assume that a nonzero vector $b$ may be written as $b = b_R + b_N$ such that $b_R \in$ range($A$) and $b_N \in$ null($A^T$), for some nonzero matrix $A$.

(a) Show that $b_R$ and $b_N$ are unique.

Assume $b_R$ and $b_N$ are not unique. Then there exist vectors $\tilde{b}_N \in$ null($A^T$) and $\tilde{b}_R \in$ range($A$) such that $b = \tilde{b}_R + \tilde{b}_N$. Then,

$$b = \tilde{b}_R + \tilde{b}_N = b_R + b_N \implies 0 = (b_R - \tilde{b}_R) + (b_N - \tilde{b}_N).$$

Since both null($A^T$) and range($A$) are subspaces, we must have $(b_R - \tilde{b}_R) \in$ range($A$) and $(b_N - \tilde{b}_N) \in$ null($A^T$). Since the zero vector is the only vector in both null($A^T$) and range($A$), $b_R - \tilde{b}_R = 0$ and $b_N - \tilde{b}_N = 0$, as required.

(b) Show that $b_R$ and $b_N$ are orthogonal.

By definition, there exists a nontrivial $v$ such that $b_R = Av$. Then $b_R^Tv = (Av)^Tv = v^TA^Tb_N = 0$. 

Since both null($A^T$) and range($A$) are subspaces, we must have $(b_R - \tilde{b}_R) \in$ range($A$) and $(b_N - \tilde{b}_N) \in$ null($A^T$). Since the zero vector is the only vector in both null($A^T$) and range($A$), $b_R - \tilde{b}_R = 0$ and $b_N - \tilde{b}_N = 0$, as required.
(c) Assuming both are nonzero, show that \( b_R \) and \( b_N \) are linearly independent.

To show that \( b_R \) and \( b_N \) are linearly independent, we write

\[
\alpha_R b_R + \alpha_N b_N = 0.
\]

(We want to show that \( \alpha_R \) and \( \alpha_N \) must be zero.)

If \( b_R \neq 0 \), multiplying by \( b_R^T \) and using the fact that \( b_R^T b_N = 0 \) gives

\[
\alpha_R b_R^T b_R = 0,
\]

which implies that \( \alpha_R = 0 \). Using a similar process when \( b_N \) is nonzero, we conclude that \( \alpha_N = 0 \). Thus, \( b_R \) and \( b_N \) must be linearly independent.

Exercise 3.3. Consider the linear system of equations \( Ax = y \) for some \( m \times n \) matrix \( A \).

(a) Assume \( y = y_R + y_N \) with \( y_R \) in \( \text{range}(A) \) and \( y_N \in \text{null}(A^T) \). Write down a condition on the range- or null-space portion of \( y \) such that \( Ax = y \) is not compatible.

For \( Ax = y \) to be compatible, \( y \) must be completely in the range of \( A \). If \( y = y_R + y_N \), then we must have \( y_N = 0 \).

(b) What can you say about the dimension of \( \text{null}(A) \) if \( Ax = y \) is compatible and has infinitely many solutions? What can you conclude about the rank of \( A \) in this case?

If \( Ax = y \) has infinitely many solutions, then the nullspace of \( A \) must be nontrivial, i.e., that its dimension is greater than 0.

We know that \( \mathbb{R}^n = \text{range}(A^T) \oplus \text{null}(A) \). Then \( n = \text{rank}(A) + \text{dim}(\text{null}(A)) \). Therefore, using the first part of this problem, we can conclude that \( \text{rank}(A) < n \) (that \( A \) has linearly dependent columns).

Exercise 3.4. Consider the matrix \( A \)

\[
A = \begin{pmatrix}
1 & -2 & -1 & 3 \\
2 & -4 & -2 & 6 \\
0 & 2 & 0 & -3
\end{pmatrix}.
\]

(This problem should be possible by inspection, without the use of Julia or Matlab.)

(a) What is the rank of \( A \)?

\( \text{rank}(A) = 2 \).

(b) Write down a basis for \( \text{range}(A) \) and \( \text{range}(A^T) \).

The range of \( A \) is the span of the columns of the matrix \( A \) (this is equivalent to the set of all \( y \) where \( Ax = y \) is compatible).

On inspections, we can see that the first and third columns of \( A \) are multiples, as are the second and fourth. We can therefore define the range of \( A \) as

\[
\text{range}(A) = \{ \alpha \begin{pmatrix}
1 \\
2 \\
0
\end{pmatrix} + \beta \begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix} : \alpha, \beta \in \mathbb{R} \}.
\]

Similarly, the range of \( A^T \) is the span of the rows of \( A \). Thus,

\[
\text{range}(A^T) = \{ \alpha \begin{pmatrix}
1 \\
-2 \\
-1
\end{pmatrix} + \beta \begin{pmatrix}
0 \\
2 \\
3
\end{pmatrix} : \alpha, \beta \in \mathbb{R} \}.
\]
(c) Define a nonzero vector $c$ such that $A^Ty = c$ is incompatible and briefly explain why the system is incompatible.

We know that if $A^Ty = c$ to be incompatible, then $c \notin \text{range}(A^T)$. Using part (b), we can choose any nonzero vector $c$ that is not a linear combination of $(1 -2 -1 3)^T$ and $(0 2 0 -3)^T$.

One possibility is

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$