Exercise 4.1. Let $A$ be an $m \times n$ matrix. Show that $c^T p = 0$ for all $p \in \text{null}(A)$ if and only if $c \in \text{range}(A^T)$. We first prove that if $c^T p = 0$ for all $p \in \text{null}(A)$, then $c$ must be in the range of $A^T$. Suppose $c = c_R + c_N$, where $c_R \in \text{range}(A^T)$ and $c_N \in \text{null}(A)$. As $c_N \in \text{null}(A)$, it must hold that $c^T c_N = 0$ leading to
\[ 0 = c^T c_N = (c_R + c_N)^T c_N = c_R^T c_N. \]
Therefore, $c_N$ must equal zero and $c = c_R \in \text{range}(A^T)$.

Now suppose $c \in \text{range}(A^T)$ so $c = A^T y$ for some vector $y$. Let $p \in \text{null}(A)$. Then,\[ c^T p = (A^T y)^T p = y^T A p = 0. \]

Exercise 4.2. Show that $F = \{ x : Ax = b \}$ is empty or convex.

If $F$ is empty, the result holds trivially.

Assume $F$ is not empty. Let $x_1$ and $x_2$ be vectors in $F$. Then we must have $Ax_1 = b$ and $Ax_2 = b$. Consider the convex combination of $x_1$ and $x_2$. Let $\theta$ be some scalar between 0 and 1. Then
\[ A(\theta x_1 + (1 - \theta)x_2) = \theta Ax_1 + (1 - \theta) Ax_2 = \theta b + (1 - \theta)b = b. \]
Thus, $\theta x_1 + (1 - \theta)x_2 \in F$ and $F$ is convex.

Exercise 4.3. Let $A$ be an $m \times n$ matrix with $\text{rank}(A) = n$ and let $b \in \text{range}(A)$. Show that the equality-constrained LP: minimize $c^T x$ subject to $Ax = b$ has a unique bounded solution, regardless of $c$. Do Lagrange multipliers exist? If so, are they unique?

If $b \in \text{range}(A)$, then $Ax = b$ is a compatible system of equations. Furthermore, since $\text{rank}(A) = n$, the columns of $A$ are linearly independent and solutions must be unique. As the feasible region of the LP consists of a single point, that single point must be the unique and bounded solution to the LP.

Lagrange multipliers $y$ must satisfy the equation $c = A^T y$. As $\text{rank}(A) = n$, we know that the nullspace of $A$ is trivial and $\mathbb{R}^n = \text{range}(A^T)$, so for any $c \in \mathbb{R}^n$, $c = A^T y$ is compatible. Thus, Lagrange multipliers must exist. The Lagrange multipliers are not unique unless $A^T$ has full column rank (in this case, when $m = n$).

Exercise 4.4. Let $C$ be an $m \times n$ matrix and let $a$ be a vector that is independent of the rows of $C$. Let $A$ be the $(m+1) \times n$ matrix
\[ A = \begin{pmatrix} C \\ a^T \end{pmatrix}. \]
Show that $Ax = e_{m+1}$ is compatible, where $e_{m+1}$ is an $(m+1)$-vector of zeros with a 1 in the $(m+1)$-th position. We make no assumptions on the rank of $C$ or the relative sizes of $m$ and $n$.

We construct a solution to the system $Ax = e_{m+1}$ to show that it is compatible.

We write $a$ in its range- and null-space portions with respect to the matrix $C$ such that $a = a_R + a_N$, with $a_R \in \text{range}(C^T)$ and $a_N \in \text{null}(C)$. As $a$ is independent of the rows of $C$, we must have $a_N \neq 0$. 

Homework Assignment #4
Due Friday February 15, 2019 by 11pm

This assignment should be submitted via gradescope.com
The programming part of the assignment is available on datahub.ucsd.edu.
Consider \( x = \frac{1}{a_N^Ta_N}a_N \in \text{null}(C) \). Then \( Cx = 0 \), so \( x \) satisfies the first \( m \) equations of \( Ax = e_{m+1} \). We consider the \((m+1)\)th equation here:

\[
a^Tx = \frac{1}{a_N^Ta_N}a_N = \frac{1}{a_N^Ta_N}a_N^Ta_N = 1.
\]

Therefore our chosen \( x \) is a solution of \( Ax = e_{m+1} \). (This solution comes from our proof of the existence of a vertex).

**Exercise 4.5.** Let \( A_a \) be a nonzero \( m \times n \) matrix, and let \( C_N \) denote the dual cone

\[
C_N = \{ w : w = A_a^T\lambda, \ \lambda \geq 0 \}.
\]

(a) Show that \( C_N \) is a convex set.

Let \( w_1 \) and \( w_2 \) be in \( C_N \). Then there exist \( y_1 \geq 0 \) and \( y_2 \geq 0 \) such that

\[
w_1 = A_a^Ty_1 \quad \text{and} \quad w_2 = A_a^Ty_2.
\]

Let \( \theta \) be a scalar between 0 and 1. Then,

\[
\theta w_1 + (1 - \theta)w_2 = \theta A_a^Ty_1 + (1 - \theta)A_a^Ty_2 = A_a^T(\theta y_1 + (1 - \theta)y_2).
\]

Clearly, \( \theta y_1 + (1 - \theta)y_2 \geq 0 \) as \( \theta \in [0, 1] \), \( \theta y_1 \geq 0 \) and \( (1 - \theta)y_2 \geq 0 \). Thus, \( \theta w_1 + (1 - \theta)w_2 \in C_N \) and \( C_N \) is a convex set.

(b) Is \( C_N \) a subspace of \( \mathbb{R}^n \)? Explain why or why not.

We define a counterexample of a dual cone that is not a subspace of \( \mathbb{R}^n \). Suppose \( A_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then the vector \( w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A_a^Ty \), with \( y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). For \( C_N \) to be a subspace, the set must be closed under scalar multiplication and addition. However, if we consider \( -w \), we have

\[
w = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = A_a^Tz, \quad \text{where} \quad z = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.
\]

Notice that \( z \) is the unique solution to the equation so \( z \) cannot be written as a nonnegative combination of the rows of \( A_a \). This \( -w \notin C_N \), and this dual cone is not a subspace of \( \mathbb{R}^2 \).

However, notice that if \( A_a = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & -3 \end{pmatrix} \), then the dual cone is equal to \( \mathbb{R}^2 \), and the dual cone would be a subspace of itself.
Problems below do not need to be turned in. Problems with an asterisk should be done in Julia.

Exercise 4.6. Let $A$ be an $m \times n$ matrix.

(a) Assume $A$ has full row rank. Show that
   
   i. $Ax = b$ is compatible for all $b \in \mathbb{R}^m$
   
   ii. $A^Ty = c$ has a unique solution if $c \in \text{range}(A^T)$

(b) Assume $A$ has full column rank. Show that
   
   i. $Ax = b$ has a unique solution if $b \in \text{range}(A)$
   
   ii. $A^Ty = c$ is compatible for all $c \in \mathbb{R}^n$

Exercise 4.7. Let $A$ be an $m \times n$ matrix. Suppose $c \notin \text{range}(A^T)$ so that $c = c_R + c_N$, with $c_N \neq 0$, $c_R \in \text{range}(A^T)$, $c_N \in \text{null}(A)$. Show that $p = -c_N$ is a descent direction for $c^T x$.

$p$ is a descent direction if $c^T p < 0$. Then

$$c^T p = (c_R + c_N)^T p = c_R^T p + c_N^T p = c_R^T (-c_N) + c_N^T (-c_N) = -c_N^T c_N.$$ 

Note that $c_R^T c_N = 0$ as the vectors lie in orthogonal subspaces. Furthermore, as $c \notin \text{range}(A^T)$, we must have $c_N \neq 0$. Thus, $c^T p = -c_N^T c_N = -\|c_N\|^2 < 0$ since $\|c_N\|^2 > 0$.

Exercise 4.8.* For each LP defined below, determine if the LP: minimize $c^T x$ subject to $Ax = b$ has a solution. If not, explain why. If so, find the minimizer $x^*$ and the Lagrange multiplier $\lambda^*$ and explain whether they are unique or not.

In the case of equality-constrained LPs, $x^*$ is an optimal solution if and only if $x^*$ is feasible and $c = A^T \lambda$ for some vector $\lambda$. (Equivalently, any feasible point is a minimizer if and only if $c = A^T \lambda$ is compatible). For each of these problems, we need to determine (1) whether feasible point(s) exist, and (2) whether Lagrange multipliers exist.

This problem will rely on Exercise 4.6 to determine existence or uniqueness of solutions to linear systems.

(a) $A = \begin{pmatrix} -4 & 1 & 3 & 2 \\ -3 & -8 & 2 & 6 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, and $c = \begin{pmatrix} -1 \\ 9 \\ 1 \\ -4 \end{pmatrix}$.

(Updated 2/26) First check the feasible region of the problem (i.e., is $Ax = b$ compatible?). Either in Julia or by inspection, you can verify that the rank of $A$ is 2, so $A$ has full row rank. Thus, $Ax = b$ is compatible for any vector $b$, and feasible points exist.

We then need to determine whether $c \in \text{range}(A^T)$ (whether Lagrange multipliers exist). In Julia, we can check the rank of $A^T$ and $(A^T c)$. If the ranks are equal, then $c \in \text{range}(A^T)$. Otherwise, $c \notin \text{range}(A^T)$.

In this case, rank($(A^T c)) = 2 = \text{rank}(A^T)$ so $c = A^T \lambda$ is compatible and Lagrange multipliers do exist. Since the system has full column rank, the multipliers are unique and we can use “backslash” to compute them:

```
julia> At = [-4 -3; 1 -8; 3 2; 2 6];
julia> c = [-1; 9; 1; -4];
julia> y = At \ c
```

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2-element Array{Float64,1}:
1.0
-1.0000000000000002

julia> At*y - c
4-element Array{Float64,1}:
8.881784197001252e-16
1.7763568394002505e-15
-4.440892098500626e-16
-1.7763568394002505e-15

(b) \[ A = \begin{pmatrix} 4 & -2 & 1 \\ 3 & 1 & 2 \\ -2 & 6 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} 10 \\ 0 \\ 5 \end{pmatrix}. \]

We can use Julia to verify the following:

julia> A = [ 4 -2 1; 3 1 2; -2 6 2];

julia> b = [1; 2; 2];

julia> c = [ 10; 0; 5];

julia> rank(A)
2

julia> rank([A b])
2

julia> At = Matrix(transpose(A))
3x3 Array{Int64,2}:
4 3 -2
-2 1 6
1 2 2

julia> rank([At c])
2

From these results, we can conclude that both \( Ax = b \) and \( c = A^T \lambda \) are compatible. Thus, any feasible point \( x \) will be a solution for this LP. Furthermore, as \( A \) is rank deficient, the nullspace of \( A \) and the nullspace of \( A^T \) are nontrivial, thus neither the feasible points nor the Lagrange multipliers are unique.

To find one particular solution, we try to find a feasible point for \( Ax = b \). As the system is singular (rank deficient), we cannot use \texttt{backslash} in Julia. What we can do is try find a solution using two linearly independent columns of \( A \). We note that the first two columns of \( A \) are linearly independent and we can find a solution \( x \) such that \( x_3 = 0 \).

julia> x = zeros(3);

julia> x[1:2] = A[:,[1,2]] \ b
2-element Array{Float64,1}:
0.4999999999999998
0.5000000000000002

julia> A*x - b
3-element Array{Float64,1}:
Thus a solution to this LP is $x^* = (0.5 \quad 0.5 \quad 0)^T$.

Similarly for Lagrange multipliers,

```julia
julia> y = zeros(3);

julia> y[1:2] = At[:,[1,2]] \ c
2-element Array{Float64,1}:
0.9999999999999997
2.000000000000001
```

```julia
julia> At*y - c
3-element Array{Float64,1}:
1.7763568394002505e-15
1.5543122344752192e-15
1.7763568394002505e-15
```

Thus, $\lambda^* = (1 \quad 2 \quad 0)^T$.

(c) $A = \begin{pmatrix} 1 & 3 & 1 \\ 9 & 7 & 1 \end{pmatrix}$, \quad $b = \begin{pmatrix} 14 \\ 2 \\ -2 \end{pmatrix}$, \quad and \quad $c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

In Julia, we check the existence of feasible points for $Ax = b$.

```julia
julia> b = [14; 2; -2];

julia> c = [5;3];

julia> A = [1 9; 3 7; 1 1];

julia> rank(A)
2

julia> rank([A b])
2
```

Thus, $Ax = b$ is compatible. Furthermore, $A$ has full column rank so the solution is unique. Using Julia,

```julia
julia> x = A\b
2-element Array{Float64,1}:
-4.000000000000001
2.000000000000001
```

For the Lagrange multipliers,

```julia
julia> At = Matrix(transpose(A))
2x3 Array{Int64,2}:
1 3 1
9 7 1

julia> rank([At c])
2
```
$c = A^T \lambda$ is compatible with infinitely many solutions (full row rank). Thus any feasible point is a minimizer of the LP. As the system is compatible, we can still use Julia to find a set of Lagrange multipliers:

```
julia> y = At\c
3-element Array{Float64,1}:
-1.1166666666666671
 1.7333333333333343
 0.9166666666666674
```
(You can also find basic solutions if you want).

(d) The same matrix $A$ and vector $b$ as in part (a), and $c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

From part (a), we know that $Ax = b$ is compatible and feasible points exist. We now need to check the existence of Lagrange multipliers.

```
julia> At = Matrix(transpose(A))
4x2 Array{Int64,2}:
-4  -3
 1  -8
 3   2
 2   6

julia> rank([At c])
3

julia> rank(A)
2
```
Since $c = A^T \lambda$ is incompatible, we conclude that this LP has no solution (in fact, the LP is unbounded).

(e) The same matrix $A$ and vector $c$ as in part (c), and $b = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$.

This problem has no feasible points

```
julia> A = [ 1 9; 3 7; 1 1];
julia> c = [5;3];
julia> b = [ 6; 3; 1];
julia> rank([A b])
3

julia> rank(A)
2
```
Thus this LP is infeasible.