Exercise 5.1. (Updated 2/19) Let $A_a$ be an $m_a \times n$ matrix of active constraints at a degenerate vertex. Let $s$ be an integer $1 \leq s \leq m_a$ such that the $s$th row of $A_a$ is dependent on the other rows of $A_a$. Show that the equations $A_ap = e_s$ are not compatible.

Let $\hat{A}_a$ denote the rows of $A_a$ with the $s$th row removed. By assumption, the $s$th row can be written as a linear combination of the other rows. Thus, we have $a_s = \hat{A}_a^T u$ for some nonzero vector $u$.

The proof is by contradiction. Suppose $p$ is the solution of $A_ap = e_s$ ($p$ must be unique because $A_a$ has full column rank). Then $\hat{A}_ap = 0$ and $1 = a_s^T p = u^T \hat{A}_ap = 0$, which is a contradiction. It follows that $A_ap = e_s$ cannot be a compatible system of equations.

Exercise 5.2. Suppose that the objective vector of a linear program is $c = (2 \quad 1)^T$, and that the matrix of active constraints at a feasible point $\bar{x}$ is

$$A_a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -3 \\ 1 & -2 \end{pmatrix}$$

(a) Is $\bar{x}$ a vertex? If so, is it degenerate or nondegenerate? Explain your answer.

The point $\bar{x}$ is a vertex because $A_a$ has rank $n = 2$. As more than two constraints are active ($A_a$ is $4 \times 2$), the vertex is degenerate.

(b) Graph the dual cone. Determine whether $\bar{x}$ is optimal. If it is optimal, compute nonnegative Lagrange multipliers.

(graph coming soon. sorry!)

The vector $c$ satisfies $c \in C_N$ and there is no direction $p$ such that $c^T p < 0$ and $A_ap \geq 0$. It follows that $\bar{x}$ is optimal.

As $\bar{x}$ is optimal, Carathéodory’s theorem tells us that at least one of the basic solutions to $c = A_a^T \lambda_a$ will give us a vector of nonnegative Lagrange multipliers.

First attempt: $W_1 = \{1, 2\}$

$$c = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = A_1^T \lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \lambda_1$$

The solution (either by hand or by Julia) is $\lambda_1 = (-1 \quad 2)^T$, so we need to try to find another basic solution.

Second attempt: $W_2 = \{1, 3\}$

$$c = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = A_2^T \lambda_2 = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \lambda_2$$

The solution is $\lambda_2 = (7 \quad 2)^T$. For this working set, $\lambda_2 \geq 0$ so we can stop now. Our “full” vector of Lagrange multipliers is then $\lambda = (7 \quad 0 \quad 2 \quad 0)^T$.

(Note that this answer is not unique! There are other possibilities depending on your choice of working set).
Exercise 5.3.* Consider the following LP:

\[
\begin{align*}
\text{minimize} \quad & 3x_1 - x_2 + 2x_3 \\
\text{subject to} \quad & -2x_1 + 4x_2 + 4x_3 \geq 6 \\
& x_1 + 4x_2 + x_3 \geq 5 \\
& -2x_1 + x_2 + 2x_3 \geq 1 \\
& 2x_1 - 2x_2 \geq 0 \\
& -3x_2 + x_3 \geq -2 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0,
\end{align*}
\]

and the point \( \bar{x} = (1, 1, 1)^T \). Find the active set at \( \bar{x} \) and determine if the point \( \bar{x} \) is optimal. If \( \bar{x} \) is not optimal, find a direction \( p \) such that

\[ c^T p < 0 \quad \text{and} \quad A_a p \geq 0. \]

We must check the residual at \( \bar{x} \) to determine the active set:

\[
\text{julia> } A = \begin{bmatrix} -2 & 4 & 4; \\
1 & 4 & 1; \\
-2 & 1 & 2; \\
2 & -2 & 0; \\
0 & -3 & 1; \\
1 & 0 & 0; \\
0 & 1 & 0; \\
0 & 0 & 1 \end{bmatrix}; \\
\text{julia> } b = \begin{bmatrix} 6; 5; 1; 0; -2; 0; 0; 0 \end{bmatrix}; \\
\text{julia> } c = \begin{bmatrix} 3; -1; 2 \end{bmatrix}; \\
\text{julia> } x = \begin{bmatrix} 1; 1; 1 \end{bmatrix};
\]

\[
\text{julia> } A \times x - b \\
8\text{-element Array}\{\text{Int64},1\}: \\
0 \\
1 \\
0 \\
0 \\
0 \\
1 \\
1 \\
1
\]

From the residual, we see that the active set consists of constraints 1, 3, 4, and 5, i.e., \( A = \{1, 3, 4, 5\} \).

\[
\text{julia> } \text{active}=\begin{bmatrix} 1,3,4,5 \end{bmatrix}; \\
\text{julia> } Aa = A[\text{active},:] \\
43\text{ Array}\{\text{Int64},2\}: \\
-2 & 4 & 4 \\
-2 & 1 & 2 \\
2 & -2 & 0 \\
0 & -3 & 1
\]

\[
\text{julia> } \text{rank}(Aa) \\
3
\]

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As there are 4 active constraints and $A_a$ has rank 3, $\bar{x}$ is a degenerate vertex and we must rely on Carathéodory’s theorem to determine optimality. There are at most 4 basic solutions (“4 choose 3”). We define each working set and compute the associated Lagrange multipliers. (Note how the indices in the working set are relative to the active set).

```julia
julia> working = [1,2,3];

julia> Aa[working,:]
3-element Array{Float64,1}:
  0.75
  -0.5
  1.75

julia> working = [1,2,4];

julia> Aa[working,:]
3-element Array{Float64,1}:
  1.7222222222222228
  -3.2222222222222228
  1.5555555555555556

julia> working = [1,3,4];

julia> Aa[working,:]
3-element Array{Float64,1}:
  0.5714285714285716
  2.0714285714285716
  -0.2857142857142857

julia> working = [2,3,4];

julia> Aa[working,:]
3-element Array{Float64,1}:
  1.5999999999999996
  3.0999999999999996
  -1.2
```

With all 4 working set choices, the multiplier vector has a negative component. Thus, we can conclude that $\bar{x}$ is not optimal.

To find a feasible descent direction, we use the solutions generated in the first part and attempt to move off constraints with a negative Lagrange multiplier. Let’s try it with working set $W = \{2,3,4\}$. We saw from above that $s = 3$ contained the negative multiplier.

```julia
julia> working = [2,3,4];

julia> Aw = Aa[working,:];

julia> es = zeros(3); es[3] = 1;

julia> p = Aw\es
3-element Array{Float64,1}:
  -0.4
  -0.4
  -0.2
```
This isn’t a feasible direction! $p$ is not feasible with respect to the active constraint that we did not include in our working set (in this case, constraint #1).

Let’s repeat the process with $W = \{1, 2, 3\}$. From our previous work, the second element of the multipliers is negative so we try to compute a direction moving off the second working constraint.

```
julia> working = [1,2,3];

julia> Aw = Aa[working, :];

julia> es = zeros(3); es[2] = 1;

julia> p = Aw\es
```

3-element Array{Float64,1}:
-0.5
-0.5
0.24999999999999997

We must check if $p$ is a feasible direction ($A_s p \geq 0$):

```
julia> Aa*p
```

4-element Array{Float64,1}:
-1.1102230246251565e-16
1.0
0.0
1.75

Great! $p$ is a feasible direction. In theory, $p$ is a descent direction ($c^T p < 0$), and in fact, should equal our negative multiplier. Let’s check it just in case:

```
julia> dot(c,p)
```

-0.5

Exercise 5.4. Consider the $k$th iteration of the simplex method.

(a) Show that the matrix defined by replacing the $s$th row of $A_k$ by the normal of a blocking constraint is nonsingular.

Let $\tilde{A}_k$ denote the $(n-1) \times n$ matrix formed from all the rows of $A_k$ except the $s$th row. As $A_k p_k = e_s$ by definition, it holds that $\tilde{A}_k p_k = 0$. Let $a_i^T x \geq \beta_i$ denote the blocking constraint, which must satisfy $a_i^T p_k < 0$.

If the matrix formed by replacing the $s$th row of $A_k$ by $a_i^T$ is singular, then $a_i$ can be written as a linear combination of the columns of $\tilde{A}_k^T$, i.e., $a_i = \tilde{A}_k^T z$, for some $z \in \mathbb{R}^{n-1}$. But then $a_i^T p_k = z^T \tilde{A}_k p_k = z^T (A_k p_k) = 0$, which cannot hold because $a_i^T p_k < 0$ for a blocking constraint. Hence we have a contradiction, and it follows that the matrix formed by replacing the $s$th row of $A_k$ by $a_i^T$ is nonsingular.
(b) Show that the component of the Lagrange multiplier vector at $x_{k+1}$ corresponding to the “new” constraint in the working set must be positive. (This implies that it is impossible to delete the constraint that was just added.)

For this solution, assume that the $s$th row of $A_k$ was replaced by the blocking constraint normal $a_t^T$, where $a_t^T$ is the $t$-th row of $A$. At $x_{k+1}$, we have $A_{k+1}^T \lambda_{k+1} = c$. Multiplying both sides of $A_{k+1}^T \lambda_{k+1} = c$ by $p_k^T$ gives

$$c^T p_k = \lambda_{k+1}^T A_{k+1} p_k.$$ 

As $p_k$ is constructed to be orthogonal to all rows of $A_k$ (and hence $A_{k+1}$) except the $s$th, we have $A_{k+1} p_k = (a_t^T p_k) e_s$, and

$$c^T p_k = (a_t^T p_k) \lambda_{k+1}^T e_s = (a_t^T p_k) (\lambda_{k+1})_s.$$ 

As $a_t^T x \geq \beta_t$ is the blocking constraint, it must satisfy $a_t^T p_k < 0$. Moreover, by definition, $(\lambda_k)_s$ was chosen to make $c^T p_k < 0$. Hence, comparing the signs of the left- and right-hand side gives $(\lambda_{k+1})_s > 0$, as required.
Problems below do not need to be turned in. Problems with an asterisk should be done in Julia.

Exercise 5.5.* Consider the linear program minimize_{x \in \mathbb{R}^n} c^T x subject to Ax \geq b. Suppose that the objective vector is \( c = (0, 2, 2)^T \), and that at a vertex \( \bar{x} \), the active constraint matrix is given by

\[
A_a = \begin{pmatrix}
1 & 2 & 3 \\
0 & 2 & 1 \\
0 & 0 & 1 \\
1 & 2 & -1 \\
0 & -1 & 0
\end{pmatrix}.
\]

(a) Is \( \bar{x} \) a degenerate or nondegenerate vertex? Explain your answer.

The vertex is degenerate since more than \( n = 3 \) constraints are active at \( \bar{x} \). This tells us that the Lagrange multipliers associated with the vertex are not unique.

(b) Verify that \( \bar{\lambda} = (\frac{1}{4}, 1, 0, -\frac{1}{4}, 0)^T \) is a vector of Lagrange multipliers at \( \bar{x} \). Is \( \bar{x} \) optimal? Explain your answer.

The product \( A_a^T \bar{\lambda} \) is given by:

\[
A_a^T \bar{\lambda} = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
2 & 2 & 0 & 2 & -1 \\
3 & 1 & 1 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\frac{1}{4} \\
1 \\
0 \\
-\frac{1}{4} \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
2 \\
2
\end{pmatrix} = c.
\]

Hence \( \bar{\lambda} \) is a Lagrange multiplier. As the fourth component of \( \bar{\lambda} \) is negative, \( \bar{\lambda} \) does not imply optimality. But from part (a), we know that \( \bar{\lambda} \) is not unique. A short survey of the basic solutions of the equation \( A_a^T \lambda = c \) reveals that the working set \( \{2, 3, 5\} \) produces an appropriate set of nonnegative Lagrange multipliers \( \lambda^* = (0, 1, 1, 0)^T \). As \( \lambda^* \geq 0 \), we conclude that \( \bar{x} \) is optimal.

Exercise 5.6. In class, we executed one iteration of the simplex method for the following LP starting at the vertex \( x_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \):

\[
\text{minimize } 2x_1 + x_2 \text{ subject to the constraints:}
\]

- constraint #1: \( x_1 + x_2 \geq 1 \)
- constraint #2: \( x_2 \geq 0 \)
- constraint #3: \( x_1 \geq 0 \)
- constraint #4: \( -x_1 \geq -2 \)
- constraint #5: \( -x_1 + x_2 \geq -2 \)

In matrix form, minimize \( c^T x \) subject to \( Ax \geq b \):

\[
c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \end{pmatrix}
\]

After one iteration of the simplex method, we arrived at the adjacent vertex \( x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) with a working set of \( \mathcal{W}_1 = \{2, 1\} \). Execute another iteration of the simplex method starting at \( x_1 \) (you should get to the optimal solution after this!).