Exercise 7.1. If $A$ and $C$ are nonsingular matrices, show that the matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is nonsingular.

Define

$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$ 

Let $u$ and $v$ be vectors such that

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

Then $Cv = 0$ and $Au + Bv = 0$. As $C$ is nonsingular, its columns are linearly independent and so $v = 0$ and $Au = 0$. Similarly, the columns of $A$ are linearly independent and $u = 0$. We have shown that $X$ has only the trivial null space and hence has linearly independent columns. As $X$ is square, it must be nonsingular.

Exercise 7.2. Write down the optimality conditions for each of the following linear programs in complementary slackness form.

(a) minimize $c^T x$ subject to $Ax \geq b$

$x^*$ is a minimizer of $c^T x$ subject to $Ax \geq b$ if and only if

- $x^*$ is feasible, i.e., $Ax^* \geq b$,
- there exist Lagrange multipliers $\lambda$ such that $c = A^T \lambda$, $\lambda \geq 0$, and
- $\lambda \cdot (Ax^* - b) = 0$.

(b) minimize $c^T x$ subject to $Ax = b$, $Dx \geq f$

$x^*$ is a minimizer of $c^T x$ subject to $Ax = b$, $Dx \geq f$ if and only if

- $x^*$ is feasible, i.e., $Ax^* = b$ and $Dx^* \geq f$,
- there exist Lagrange multipliers $\pi^*$ and $\lambda^*$ such that $c = A^T \pi^* + D^T \lambda^*$ with $\lambda^* \geq 0$, and
- $\lambda^* \cdot (Dx^* - f) = 0$.

(c) minimize $c^T x$ subject to $Ax = b$, $x \geq 0$

$x^*$ is a minimizer of $c^T x$ subject to $Ax = b$, $x \geq 0$ if and only if

- $x^*$ is feasible, i.e., $Ax^* = b$ and $x^* \geq 0$,
- there exist Lagrange multipliers $\pi^*$ and reduced costs $z^*$ such that $c = A^T \pi^* + z^*$ with $z^* \geq 0$, and
- $z^* \cdot x^* = 0$.

Exercise 7.3. Consider the standard-form problem of minimizing $c^T x$ subject to $Ax = b$, $x \geq 0$, with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix}.$$
(a) Use any method of your choice to find a vertex for this constraint set.

A vertex is a nonnegative basic solution of $Ax = b$. The basis $B = \{1, 3\}$ gives the basis matrix

$$ B = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}. $$

This basis gives the vector $x_B$ such that $Bx_B = b$ as

$$ x_B = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{so that} \quad x = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}. $$

Other vertices are defined by bases $B = \{2, 3\}$ and $B = \{3, 4\}$.

(b) Is the vertex optimal? Explain why or why not.

The $\pi$-values for the basis $B = \{1, 3\}$ satisfy $B^T \pi = c_B$, i.e.,

$$ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \pi = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \implies \pi = \begin{pmatrix} -5 \\ 4 \end{pmatrix}. $$

The reduced costs are

$$ z = c - A^T \pi = \begin{pmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3/4 \\ 3/4 \end{pmatrix} \geq 0. $$

The reduced costs are nonnegative, which implies that this is an optimal vertex. The objective value is $-\frac{7}{4}$. The objective functions at the basic solutions $B = \{2, 3\}$ and $B = \{3, 4\}$ are $-1$ and $-\frac{8}{3}$.

Exercise 7.4.* Consider the linear program

$$ \begin{align*} 
\text{minimize} & \quad x_2 - x_3 - 2x_4 \\
\text{subject to} & \quad x_2 + 2x_3 \leq 1 \\
& \quad x_1 + x_2 + x_3 + x_4 \leq 1 \\
& \quad x_1 - x_2 - 5x_3 + 3x_4 \leq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_3 \geq 0 \\
& \quad x_4 \geq 0.
\end{align*} $$

(a) Convert this problem into standard-form $\min c^T x$ subject to $Ax = b, x \geq 0$.

We add three slack variables $x_5, x_6,$ and $x_7$, giving

$$ c = \begin{pmatrix} 0 \\ -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -5 & 3 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, $$

with $x \geq 0$. 

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(b) Compute one iteration of the standard-form simplex method for this problem, starting at the basic solution \( x_B = (\frac{1}{3}, \frac{1}{2}, 2)^T \) defined by columns 3, 4 and 7 of \( A \). Show your work. Be sure to write down the objective function, basic set, nonbasic set, \( \pi \)-vector and reduced costs at both the beginning and end of the iteration. Check that the new iterate is feasible, with improved objective value.

With \( x_B \) as stated, we find that \( c^T x = -\frac{3}{2} \). We start the iteration with Step 1 of Algorithm 5.1.

**Step 1:** We have \( B = \{3, 4, 7\}, N = \{1, 2, 5, 6\} \), with \( x_B = (\frac{1}{3}, \frac{1}{2}, 2)^T \). We also have

\[
B = \begin{pmatrix}
2 & 0 & 0 \\
1 & 1 & 0 \\
-5 & 3 & 1
\end{pmatrix}, \quad c_B = \begin{pmatrix}
-1 \\
-2 \\
0
\end{pmatrix}
\]

and

\[
N = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & -1 & 0 & 0
\end{pmatrix}, \quad c_N = \begin{pmatrix}
0 \\
-1 \\
0
\end{pmatrix}.
\]

Solving the (triangular) system \( B^T \pi = c_B \) gives \( \pi = (\frac{1}{4}, -2, 0)^T \). The calculation \( z_N = c_N - NT \pi \) gives \( z_N = (2, \frac{1}{2}, -\frac{1}{2}, 2)^T \). There is a negative reduced cost, so we move to the next step.

**Step 2:** We have immediately that \( s = 3 \), which gives \( \nu_s = 5 \), thus \( \alpha_s = (1, 0, 0)^T \). The solution of the system \( BP_B = -a_{\nu_s} \) is \( p_B = (-\frac{1}{2}, \frac{1}{2}, -4)^T \).

**Step 3:** The decreasing basic variables are the first and the third. The computation of \( \sigma \) is thus given by \( \sigma_1 = (1/2)/(-(-1/2)) = 1 \), \( \sigma_2 = +\infty \), and \( \sigma_3 = 2/(-(-4)) = 1/2 \). Thus we find that \( \alpha = \frac{1}{2} \). We proceed to the next step.

**Step 4:** As \( \alpha \) is bounded, we move to the next step.

**Step 5:** We have \( t = 3 \), which implies that \( \beta_t = 7 \). The computation of \( x_B \) yields \( x_B = (\frac{1}{3}, \frac{3}{2}, 0)^T \). Note that this \( x_B \) is associated with the \( B \) of Step 1. We set \( x_{\nu_s} = x_5 = \alpha = \frac{1}{2} \), \( B = \{3, 4, 5\}, N = \{1, 2, 7, 6\} \), thus

\[
B = \begin{pmatrix}
2 & 0 & 1 \\
1 & 1 & 0 \\
-5 & 3 & 0
\end{pmatrix}, \quad N = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 \\
1 & -1 & 1 & 0
\end{pmatrix}.
\]

With the change of basis, we have \( x_B = (\frac{1}{4}, \frac{3}{2}, \frac{1}{2})^T \), which is feasible with \( c^T x = -\frac{3}{2} \). We now return to Step 1.

**Step 1:** With the change in basis, we have \( c_B = (-1, -2, 0)^T \), thus the solution of the system \( B^T \pi = c_B \) is \( \pi = (0, -1\frac{2}{3}, -\frac{1}{3})^T \). Since we now have \( c_N = (0, -1, 0, 0)^T \), the computation of \( z_N \) gives \( z_N = (1\frac{2}{3}, \frac{1}{2}, \frac{1}{2}, 1\frac{2}{3})^T \). We happened upon the optimal solution in one iteration. The solution is \( x^* = (0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4})^T \) in terms of the original variables.

**Exercise 7.5.** Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad d^T w \\
\text{subject to} & \quad Gw \geq f, \quad w \geq 0,
\end{align*}
\]

where \( G \) is an \( m \times n \) matrix.

(a) Convert this problem to all-inequality form and to standard form. In both cases, write down the problem in terms of “\( x \)” for the vector of variables, “\( c \)” for the objective, and “\( A \)” for the matrix of constraints.

**The LP in all-inequality form is:**

\[
\begin{align*}
\text{minimize} & \quad d^T x \\
\text{subject to} & \quad Ax \geq b.
\end{align*}
\]
where \( x = w, \quad A = \begin{pmatrix} G \\ I \end{pmatrix} \) and \( b = \begin{pmatrix} f \\ 0 \end{pmatrix} \).

If the LP is written in standard form then:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \quad x \geq 0.
\end{align*}
\]

where \( x = \begin{pmatrix} y \\ z \end{pmatrix}, \quad b = f, \quad c = \begin{pmatrix} d \\ 0 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} G & -I_m \end{pmatrix}.\)

(b) Formulate the dual form of the linear program.

If we start from the LP in standard form, the problem has data:

\[
\begin{pmatrix}
  d^T & 0 \\
  G & -I_m \\
\end{pmatrix}
\begin{pmatrix}
  f
\end{pmatrix}.
\]

The dual is in all-inequality form, with data:

\[
\begin{pmatrix}
  -f^T \\
  -G^T \\
  -(-I_m) \\
\end{pmatrix}
\begin{pmatrix}
  -d \\
  0
\end{pmatrix}.
\]

The all-inequality-form problem is

\[
\begin{align*}
\text{minimize} & \quad (-f)^T y \\
\text{subject to} & \quad (-G^T I_m) y \geq \begin{pmatrix} -d \\ 0 \end{pmatrix}.
\end{align*}
\]

If we split the constraints, then

\[
\begin{align*}
\text{minimize} & \quad (-f)^T y \\
\text{subject to} & \quad (-G^T) y \geq -d, \quad y \geq 0.
\end{align*}
\]

Multiplying the objective and constraints by \(-1\) gives

\[
\begin{align*}
\text{maximize} & \quad f^T y \\
\text{subject to} & \quad G^T y \leq d, \quad y \geq 0.
\end{align*}
\]

We would get the same system if we started from all-inequality form.

(c) Suppose that you are given a specific problem in the above LP form. Briefly discuss the circumstances under which you would use the primal or dual form to solve the problem.

First, suppose that you want to use a simplex code that requires the constraints to be in all-inequality form. Converting the problem to all-inequality form gives the matrix

\[ A = \begin{pmatrix} G \\ I_n \end{pmatrix}, \]

which is the right shape for the simplex method in all-inequality form if \( m > n \). If \( n > m \), it is more efficient to take the dual before converting the problem to all-inequality form. In this case the constraint matrix is

\[ A = \begin{pmatrix} -G^T \\ I_m \end{pmatrix}. \]
Next, suppose that you want to use a simplex code that requires the constraints to be in standard form. Converting the problem to standard form gives the matrix
\[
A = \left( \begin{array}{c} G \quad -I_m \end{array} \right),
\]
which is the right shape for the simplex method in standard form if \( m < n \). If \( m > n \), it is more efficient to take the dual before converting the problem to standard form. In this case the constraint matrix is
\[
A = \left( \begin{array}{c} G^T \quad I_n \end{array} \right).
\]

If you have codes for both all-inequality and standard form, you would not need to take the dual because you can convert the problem to all-inequality form if \( m > n \) and standard form if \( m < n \).

**Exercise 7.6.** Formulate dual problems for the following linear programs:

(a) \[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b, \quad a^T x = \beta.
\end{align*}
\]

We write \( a^T x = \beta \) as the two inequalities \( a^T x \geq \beta \) and \( a^T x \leq \beta \), i.e.,
\[
\begin{align*}
a^T x & \geq \beta \\
-a^T x & \geq -\beta.
\end{align*}
\]

We then define
\[
\tilde{A} = \left( \begin{array}{c} A \\
a^T \\
-a^T \end{array} \right), \quad \tilde{b} = \left( \begin{array}{c} b \\
\beta \\
-\beta \end{array} \right).
\]

The primal problem is then
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \tilde{A} x \geq \tilde{b}.
\end{align*}
\]

With the primal in all-inequality form, we write the dual in standard form as:
\[
\begin{align*}
\text{maximize} & \quad \bar{b}^T \lambda \\
\text{subject to} & \quad \tilde{A}^T \lambda = c, \quad \lambda \geq 0.
\end{align*}
\]

If \( \lambda \) is partitioned as \( \lambda = (v^T, u_1, u_2)^T \), where \( v \in \mathbb{R}^m \), and \( u_1 \), and \( u_2 \) are scalars. Then
\[
\bar{b}^T \lambda = \left( \begin{array}{c} b^T \\
\beta \\
-\beta \end{array} \right) \left( \begin{array}{c} v \\
u_1 \\
u_2 \end{array} \right) = b^T v + \beta u_1 - \beta u_2 = b^T v + \beta (u_1 - u_2),
\]

and
\[
\tilde{A}^T \lambda = \left( \begin{array}{c} A^T \\
a \\
-a \end{array} \right) \left( \begin{array}{c} v \\
u_1 \\
u_2 \end{array} \right) = A^T v + au_1 - au_2 = A^T v + a(u_1 - u_2).
\]

If we define \( \pi = u_1 - u_2 \), then the final dual is
\[
\begin{align*}
\text{maximize} & \quad \bar{b}^T v + \beta \pi \\
\text{subject to} & \quad A^T v + \pi a = c, \quad v \geq 0.
\end{align*}
\]

Notice that the dual variable \( \pi \) associated with the equality constraint may be either positive or negative.
This time, we proceed by converting the primal into standard form. Assume that $A$ and $B$ are $m \times n$ and $l \times n$ respectively. We write $Bx \leq d$ as $Bx + y = d$, with $y \geq 0$. We define
\[
\tilde{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b \\ d \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} A & 0 \\ B & I \end{pmatrix}.
\]

With these definitions, the primal problem is
\[
\begin{align*}
\text{minimize} \quad & \tilde{c}^T \tilde{x} \\
\text{subject to} \quad & \tilde{A} \tilde{x} = \tilde{b}, \quad \tilde{x} \geq 0.
\end{align*}
\]

The dual is given by
\[
\begin{align*}
\text{maximize} \quad & b^T \pi \\
\text{subject to} \quad & A^T \pi \leq \tilde{c}.
\end{align*}
\]

We partition $\pi$ as
\[
\pi = \begin{pmatrix} u \\ v \end{pmatrix},
\]

then $b^T \pi = b^T u + d^T v$ and
\[
A^T \pi = \begin{pmatrix} A^T \\ B^T \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A^T u + B^T v \\ v \end{pmatrix}.
\]

The final dual is then
\[
\begin{align*}
\text{maximize} \quad & b^T u + d^T v \\
\text{subject to} \quad & A^T u + B^T v \leq c, \quad v \leq 0.
\end{align*}
\]